ON AUTOMORPHISMS OF BLOWUPS OF $\mathbb{P}^3$

TUYEN TRUNG TRUONG

Abstract. Let $\pi : X \to \mathbb{P}^3$ be a finite composition of blowups along smooth centers. We show that for "almost all" of such X, if $f \in Aut(X)$ then its first and second dynamical degrees are the same. We also construct many examples of finite blowups $X \to \mathbb{P}^3$, whose automorphism group $Aut(X)$ has only finitely many connected components.

We also present a heuristic argument showing that for a "generic" compact Kähler manifold X of dimension ≥ 3, the automorphism group $Aut(X)$ has only finitely many connected components.

1. Introduction

While there are many examples of compact complex surfaces having automorphisms of positive entropies (works of Cantat [9], Bedford-Kim [5][6][7], McMullen [21][22][23][24], Oguiso [26][27], Cantat-Dolgachev [10], Zhang [36], Diller [14], Déserti-Grivaux [13], Reschke [31],...), there are few interesting examples of manifolds of higher dimensions having automorphisms of positive entropies (Oguiso [28][29], Oguiso-Perroni [25],...). Some restrictions on projective 3-manifolds having automorphisms of positive entropies are known (Zhang [33][35],...). On blowups of $\mathbb{P}^3$ or of products of projective spaces $\mathbb{P}^k$, only pseudo-automorphims of positive entropies are constructed up to date (Bedford-Kim [4], Perroni-Zhang [30], Blanc [8],...).

This paper concerns the dynamical degrees and topological entropies of automorphisms of finite blowups $X \to \mathbb{P}^3$. (Definitions of dynamical degrees and topological entropies are given in the next section.)

Theorem 1. *Let $X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathbb{P}^3$ be a finite composition of blowups along smooth centers. Assume that each $X_{j+1} \to X_j$ ($0 \leq j \leq n-1$) is either*

1) A blowup of X_j at a point

or

2) A blowup of X_j along a smooth curve $C \subset X_j$, so that $c_1(X_j).C \neq 2(g-1)$, where $c_1(X_j)$ is the first Chern class of X_j and g is the genus of C.

Then, for every $f \in Aut(X)$, its dynamical degrees satisfy $\lambda_1(f) = \lambda_2(f)$.

If we incorporate the constructions from Theorem 2 below into Theorem 1, we see that for "almost all" finite blowups $X \to \mathbb{P}^3$, if $f \in Aut(X)$ then its first and second dynamical degrees are the same. Next we construct many examples of finite blowups $X \to \mathbb{P}^3$ so that every element of $Aut(X)$ has zero topological entropy. In

Date: December 25, 2012.

2010 *Mathematics Subject Classification.* 37F, 14D, 32U40, 32H50.

Key words and phrases. Automorphisms, Blowups of $\mathbb{P}^3$, Dynamical degrees, Topological entropy.

Corollary 1, we show that most of the examples constructed in Theorem 2 satisfies the stronger constraint that their automorphism group $Aut(X)$ have only finitely many connected components.

Theorem 2. *Let $X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathbb{P}^3$ be a finite composition of blowups along smooth centers. Assume that $X_1 \to X_0 = \mathbb{P}^3$ is a blowup of a finite number of points $p_1, \ldots, p_s \in \mathbb{P}^3$ and smooth curves $C_1, \ldots, C_t \subset \mathbb{P}^3$ which are in general positions, that is no point belongs to a curve and two distinct curves are disjoint.*

Moreover, assume that each $X_{j+1} \to X_j$ $(1 \leq j \leq n-1)$ is either

1) A finite composition of blowups, the images in X_j of the exceptional divisors are points;

or

2) A blowup of X_j along a smooth curve $C \subset X_j$ so that $\gamma = c_1(X_j).C + 2g - 2 < 0$, here $c_1(X_j)$ is the first Chern class of X_j and g is the genus of C. Moreover, assume that C is not the unique effective curve in its cohomology class;

or

3) A blowup of X_j along a smooth curve C contained in an irreducible hypersurface S of X_j so that $2\kappa < \mu\gamma$. Here $\kappa = S.C$, $1 \leq \mu =$ the multiplicty of C in S, and $\gamma = c_1(X_j).C + 2g - 2$ is the same as in 2).

Then, for every $f \in Aut(X)$, its dynamical degrees satisfy $\lambda_1(f) = \lambda_2(f) = 1$. Therefore, by Gromov-Yomdin's theorem (see Theorem 4 below), $h_{top}(f) = 0$.

As a consequence we obtain the following

Corollary 1. *Let $X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathbb{P}^3$ be a finite composition of blowups along smooth centers. Assume that each $X_{j+1} \to X_j$ $(1 \leq j \leq n-1)$ is either one of the cases i), ii), iii) in Theorem 2. Then the automorphism group $Aut(X)$ has only finitely many connected components.*

Proof. In this case we can take $X_1 = X_0 = \mathbb{P}^3$ in Theorem 2. The proof of Theorem 2 implies that if ζ is a non-zero nef class on X then $\zeta.\zeta \neq 0$. Hence the proof of Theorem 1.1 in the paper Bayraktar-Cantat [2] implies that $Aut(X)$ has only finitely many connected components. □

Remark:

-Even though we stated Theorems 1, Corollary 1 and 2 only for $\mathbb{P}^3$, we can modify them to apply to other spaces, for example $\mathbb{P}^2 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Theorem 1 and Corollary 1 can be repeated verbatim, while the conclusions of Theorem 2 still hold if we do not include the space X_1 in the statement. This is necessary, since if Z is an appropriate blowup of $\mathbb{P}^2$ at points in $\mathbb{P}^2$ then Z has an automorphism g of positive entropy (see McMullen [23]), therefore the space $Z \times \mathbb{P}^1$ has an automorphism of positive entropy as well. This space $Z \times \mathbb{P}^1$ is one of the spaces X_1 if we start from $X_0 = \mathbb{P}^2 \times \mathbb{P}^1$, yet it has an automorphism of positive entropy. We will show however that this example is compatible with our Theorem 2 in that the curves we blowup to form the space $Z \times \mathbb{P}^1$ do not satisfy both conditions 2) and 3) of Theorem 2. We will also construct blowups of pairwise disjoint curves on $\mathbb{P}^2 \times \mathbb{P}^1$ that do satisfy at least one of these conditions. For details please see the last section.

-Conditions 2) and 3) in Theorem 2 are complement to each other: 2) is applied for $\gamma < 0$ while 3) may be applied for $\gamma \geq 0$. The examples mentioned in the above paragraph show that conditions 2) and 3) are somewhat optimal, and there are cases when condition 2) does not apply while condition 3) does apply.

-Consider condition 2) in Theorem 2. Let $F \subset X_{j+1}$ be the exceptional divisor of the blowup $X_{j+1} \to X_j$, and let $M \subset F$ be a fiber of the projection $F \to C$. If there is a non-zero effective curve $V \subset F$ so that $F.V \geq 0$ then the requirement that C is not the only effective curve in its cohomology class is not needed. For further comment on this, please see Lemma 5.

-Consider condition 3) of Theorem 2. Let $F \subset X_{j+1}$ be the exceptional divisor of the blowup $X_{j+1} \to X_j$, and let $M \subset F$ be a fiber of the projection $F \to C$. Then condition 3) implies the existence of an effective curve $C_0 \subset F$ with the properties $C_0.C_0 < 0$ and $C_0.M > 0$. Part e) of the proof of of Theorem 2 and Lemma 4 implies that Theorem 2 still holds if we replace condition 3) by the latter. For further comment on this please see Lemma 5.

-In condition 3) of Theorem 2, given an irreducible curve $C \subset Y$, there is always a hypersurface $S \subset Y$ containing C. In fact, if C is in the strict transform of an exceptional divisor then we can choose S to be that hypersurface. Otherwise, C is the strict transform of some curve $D \subset \mathbb{P}^3$. In this case we choose S to be the strict transform of a hypersurface in $\mathbb{P}^3$ containing D.

-It is a natural and important problem to understand finer structures of the automorphisms of blowups of $\mathbb{P}^3$ (such as whether the automorphism groups of a generic space constructed in Theorem 2 is trivial, see also Question 1 below). Unfortunately we are not able to answer this in the current paper.

There are many examples realizing the conditions of Theorem 2 (and Theorem 1).

Example 1: For condition 1), we blowup a point in X_j and then we can blowup any number of points and curves on the exceptional divisor, and then can do iterated blowups on the resulting exceptional divisors and so on.

Example 2: For condition 2), assume that we have a smooth curve D on X_j and another effective curve D' in the cohomology class of D so that D and D' intersect in a large enough number of points (counted with multiplicities). Then we blowup these intersection points (may need to do iterated blowup when the multiplicity is greater than 1), and then blowup the strict transform of D. Another way of constructing is to blowup many curves having non-empty intersections with D (see Example 6).

Example 3: Let D be a smooth curve of degree $d \geq 2$ contained in a hyperplane W of $\mathbb{P}^3$. If $Y \to \mathbb{P}^3$ is the blowup of t points on D (for any number t), and C is the strict transform of D then condition 3) is satisfied if we choose S to be the strict transform of the hyperplane containing D. If in contrast, D has degree 1 (and therefore is a projective line), then we can apply Theorem 1 provided $t \neq 3$ (See Example 5 also).

Example 4: Let C_1 and C_2 be two smooth curves, both belonging to the same hyperplane $W \subset \mathbb{P}^3$. Let $Y \to \mathbb{P}^3$ be the blowup at C_1, and let $X \to Y$ be the blowup at the strict transform of C_2. Then any automorphism of X has zero entropy.

Example 5: Let $Y \to \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ at 4 points $e_0 = [1 : 0 : 0 : 0]$, $e_1 = [0 : 1 : 0 : 0]$, $e_2 = [0 : 0 : 1 : 0]$ and $e_3 = [0 : 0 : 0 : 1]$. For $0 \leq i \neq j \leq 3$, let $\Sigma_{i,j} \subset \mathbb{P}^3$ be the line connecting e_i and e_j. Let $\widetilde{\Sigma_{i,j}}$ be the strict transform in Y of $\Sigma_{i,j}$. The curves $\widetilde{\Sigma_{i,j}}$ are pairwise disjoint. Let $X \to Y$ be the blowup of Y along the curves $\widetilde{\Sigma_{i,j}}$. Then for every element f of $Aut(X)$ we have $\lambda_1(f) = \lambda_2(f)$. (See Example 3 also).

Example 6: Notations are as in Example 5. Let $X\rightarrow \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ at e_1 and e_3, followed by blowup of the strict transform of $\Sigma_{0,1}$ and then blowup of the strict transform of $\Sigma_{0,3}$. Then any automorphism of X has zero topological entropy. Bedford and Kim [4] constructed this space in connection with pseudo-automorphic linear fractional maps. Theorem 2 does not apply directly to this example but we can adapt the proof to it.

Theorem 2 gives support to the guess that the answer to the following question, asked by Professor Eric Bedford in a conference in Paris in Jun 2011, is No:

Question 1: Is there a finite blowup $\pi : X\rightarrow \mathbb{P}^3$ and an automorphism $f : X\rightarrow X$ with $h_{top}(f)>0$?

We end this section giving a heuristic argument to explain why there are few automorphisms of positive entropies of projective (or more generally, compact Kähler) manifolds X of dimension 3 (and higher dimensions). Let $f\in Aut(X)$ and choose η a non-zero nef-class (see the next section for definition of nef-classes) which is an eigenvector of eigenvalue $\lambda_1(f)$ of the linear map $f^*:H^{1,1}(X)\rightarrow H^{1,1}(X)$ (the existence of such an η is assured by a Perron-Frobenius type theorem, see the proof of Theorem 7). As the proof of Theorem 7 below shows, if for every $f\in Aut(X)$ we can choose such an η so that $\eta.\eta\neq 0$ then every automorphism of X has zero entropy. It is very unlikely to have $\eta.\eta=0$. In fact, by Poincare duality and Hodge decomposition, $dim(H^{1,1}(X))=dim(H^{2,2}(X))$. Denote by n this dimension, let $x_1,\ldots ,x_n$ be a basis for $H^{1,1}(X)$ and let $y_1,\ldots ,y_n$ be a basis for $H^{2,2}(X)$. Then there are numbers $a_1,\ldots ,a_n$ so that $\eta =a_1x_1+\ldots +a_nx_n$. We can write $\eta.\eta = P_1(a_1,\ldots ,a_n)y_1+\ldots +P_n(a_1,\ldots ,a_n)y_n$, here $P_1(a_1,\ldots ,a_n),\ldots ,P_n(a_1,\ldots ,a_n)$ are homogeneous polynomials of degree 2 in the variables $a_1,\ldots ,a_n$. The coefficients of these polynomials depend only on the intersection product on the cohomology groups of X. If $\eta.\eta=0$, then $P_1(a_1,\ldots ,a_n)=\ldots =P_n(a_1,\ldots ,a_n)=0$. The latter, being a system of n homogeneous equations in n variables, is expected to have only the solution $a_1=\ldots =a_n=0$, even when we do not take into account the fact that η is nef and is an eigenvector of eigenvalue $\lambda_1(f)$ of f^*.

Remarks.

1. In recent works Bayraktar [1] and Bayraktar-Cantat [2], the authors considered a more refined condition of that used in the above heuristic argument. More precisely, they considered the manifolds X such that for any non-zero nef class $\zeta\in H^{1,1}(X)$ then ζ^{k-r+1} is non-zero, here r is a fixed integer with $k>2r+2$.

2. Combined with the results in [2], the above heuristic argument can be applied to prove a stronger conclusion: For "generic" compact Kähler manifolds X of dimension ≥ 3, the automorphism group $Aut(X)$ has only finitely many connected components.

Acknowledgements. The author is grateful to Professor Tien-Cuong Dinh for suggesting that the answer to Question 1 should be No. He thanks Professor Mattias Jonsson for checking an earlier version of this paper, for his useful comments and interest in the topic of the paper; and thanks Professor Viet-Anh Nguyen for checking an earlier version of this paper and for useful discussions. He thanks Professors Ekaterina Amerik and Eric Bedford for helpful discussions. He is also thankful to Professors Laura DeMarco, Roland Roeder, and Joseph Silverman for their interest in the topic of the paper.

2. Preliminaries on positive cohomology classes, blowups, dynamical degrees, and entropies

2.1. **Kähler, nef and psef classes, and effective varieties.** Let X be a compact Kähler manifold. Let $\eta \in H^{1,1}(X)$. We say that η is Kähler if it can be represented by a Kähler $(1,1)$ form. We say that η is nef if it is a limit of a sequence of Kähler classes. We say that η is psef if it can be represented by a positive closed $(1,1)$ current. A class $\xi \in H^{p,p}(X)$ is an effective variety if there are irreducible varieties $C_1,\ldots,C_t$ of codimension p in X and non-negative real numbers $a_1,\ldots,a_t$ so that ξ is represented by $\sum_i a_iC_i$.

Demailly and Paun [12] gave a characterization of Kähler and nef classes, which in the case of projective manifolds is summarized as follows:

Theorem 3. *Let X be a projective manifold with a Kähler $(1,1)$ form ω. A class $\eta \in H^{1,1}(X)$ is Kähler if and only for any irreducible subvariety $V \subset X$ then $\int_V \eta^{dim(V)} > 0$. A class $\eta \in H^{1,1}(X)$ is nef if and only for any irreducible subvariety $V \subset X$ then $\int_V \eta^{dim(V)-j}\wedge\omega^j \geq 0$ for all $0 \leq j \leq dim(V)$.*

Nef classes are preserved under pullback by holomorphic maps.

Lemma 1. *Let $\pi : X \to Y$ be a holomorphic map between compact Kähler manifolds. Then $\pi^*(H^{1,1}_{nef}(X)) \subset H^{1,1}_{nef}(Y)$.*

Proof. Since nef classes are in the closure of Kähler classes, it suffices to show that if η is a Kähler class then $\pi^*(\eta)$ is nef. Let φ be a Kähler $(1,1)$ form representing η. Then $\pi^*(\varphi)$ is a positive smooth $(1,1)$ form. Let ω_X be a Kähler $(1,1)$ form on X. Then $\pi^*(\eta)$ is represented as a limit of the following Kähler classes

$$\pi^*(\varphi)+\frac{1}{n}\omega_X,$$

and hence is nef. □

Remark: Similarly, it can be shown that psef classes are preserved under pushforward by holomorphic maps. However, nef classes may not be preserved under pushforwards, even when the map is a blowup.

2.2. **Blowup of a projective 3-manifold at a point.** Let $\pi : X \to Y$ be the blowup of a projective 3-manifold at a point p. Let $E = \mathbb{P}^2$ be the exceptional divisor and let $L \subset E$ be a line. Then $H^{1,1}(X)$ is generated by $\pi^*(H^{1,1}(Y))$ and E, and $H^{2,2}(X)$ is generated by $\pi^*(H^{2,2}(Y))$ and L. The intersection product on the cohomology of X is given by

$$\pi^*(\xi).E = 0,\ E.E = -L,$$
$$\pi^*(\xi).L = 0,\ E.L = -1.$$

The first and second Chern classes of X can be computed by (see e.g. Section 6, Chapter 4 in the book of Griffiths-Harris [19])

$$\begin{aligned} c_1(X) &= \pi^*(c_1(Y)) - 2E,\\ c_2(X) &= \pi^*(c_2(Y)). \end{aligned}$$

The following result concerns the relations between cycles on X and Y.

Lemma 2. *For any effective curve $V \subset Y$, there is an effective curve $\widetilde{V} \subset X$ so that $\pi_*(\widetilde{V}) = V$ and $\widetilde{V}.E \geq 0$.*

Proof. It suffices to consider the case when V is an irreducible curve. We can choose $\widetilde{V}$ to be the strict transform of V. Then $\pi_*(\widetilde{V}) = V$, and $\widetilde{V}$ is not contained in E. Therefore $\widetilde{V}.E \geq 0$. □

We end this subsection showing that nef classes are preserved under pushforward by point-blowups.

Lemma 3. *Let* $\eta \in H^{1,1}_{nef}(X)$. *Then* $\pi_*(\eta) \in H^{1,1}_{nef}(Y)$.

Proof. It suffices to prove the conclusion when η is a Kähler class. Let φ be a Kähler $(1,1)$ form representing η. Then $\pi_*(\varphi)$ is a positive closed $(1,1)$ current, which is smooth on $X - p$.

Let ω_Y be a Kähler $(1,1)$ form on Y. To show that $\pi_*(\eta)$ is a nef class, by Theorem 3 it suffices to show that for any irreducible variety $V \subset Y$ then $\pi_*(\eta)^{dim(V)-j}.V.\omega_Y^j \geq 0$ for $0 \leq j \leq dim(V)$. We let $[V]$ be the current of integration on V. Then by the results in Section 4, Chapter 3 in the book of Demailly [11], the current $\pi_*(\varphi)^{dim(V)-j} \wedge [V] \wedge \omega_Y^j$ is well-defined and is a positive measure, whose mass equals to $\pi_*(\eta)^{dim(V)-j}.V.\omega_Y^j$. Thus the latter quantity is non-negative. □

2.3. **Blowup of a projective 3-manifold along a smooth curve.** Let $\pi : X \to Y$ be the blowup of a projective 3-manifold along a smooth curve $C \subset Y$. Let g be the genus of C. Let F be the exceptional divisor and let M be a fiber of the projection $F \to C$. We can identify F with the projective bundle $\mathbb{P}(\mathcal{E}) \to C$, where $\mathcal{E} = N_{C/Y} \to C$ is the normal vector bundle of C in Y.

Then $H^{1,1}(X)$ is generated by $\pi^*(H^{1,1}(Y))$ and F, and $H^{2,2}(X)$ is generated by $\pi^*(H^{2,2}(Y))$ and M. The intersection between F and M is $F.M = -1$. The first and second Chern classes of X can be computed as follows:

$$
\begin{aligned}
c_1(X) &= \pi^*(c_1(Y)) - F, \\
c_2(X) &= \pi^*(c_2(Y) + C) - \pi^* c_1(Y).F.
\end{aligned}
$$

Let $[F] \to X$ be the line bundle of F in X, and denote by $e = [F]|_F$. Then (see e.g. Section 6, Chapter 4 in the book of Griffiths - Harris [19]) in F we have the equalities

$$e.M = -1, \ e.e = -c_1(\mathcal{E}).$$

From the SES of vector bundles on C

$$0 \to T_C \to T_Y|_C \to \mathcal{E} \to 0,$$

it follows by the additivity of first Chern classes that

$$c_1(\mathcal{E}) = c_1(T_Y).C - c_1(T_C) = c_1(Y).C + 2g - 2.$$

We define

$$\gamma := c_1(Y).C + 2g - 2.$$

Since $F \to C$ is a ruled surface (i.e. its fibers are projective lines $\mathbb{P}^1$), there is a canonical section C_0 which is the image of a holomorphic map $\sigma_0 : C \to F$ (see e.g. Section 2, Chapter 5 in Hartshorne's book [20]). Therefore C_0 is an effective curve in F. Such a C_0 has intersection 1 with a fiber M.

We will return to the canonical section C_0 at the end of this subsection. For now, we however work in a more general assumption on C_0, for using later. That is, we consider an effective curve $C_0 \subset F$ with the following properties

$$\begin{aligned} C_0.C_0 &= \tau, \\ C_0.M &= \mu > 0, \\ M.M &= 0. \end{aligned}$$

Any divisor on F is numerically equivalent to a linear combination of C_0 and M. We now show the following

Lemma 4. *a)*

$$F.C_0 = \frac{1}{2}(\gamma\mu - \frac{\tau}{\mu}). \tag{2.1}$$

b)

$$F.F = -\frac{1}{\mu}C_0 + \frac{1}{2}(\frac{\tau}{\mu^2} + \gamma)M.$$

c) $\pi_*(F.F) = -C$.

Proof. a) In fact, we have

$$F.C_0 = [F]|_{C_0} = [F]|_F.C_0 = e.C_0,$$

here the two expressions on the RHS are computed in F. On F, numerically we can write $e = aC_0 + bM$. Then from $-1 = e.M = (aC_0 + bM).M = a\mu$, we get $a = -1/\mu$. Substitute this into $e.e = -\gamma$ we obtain

$$-\gamma = e.e = (\frac{1}{\mu}C_0 - bM).(\frac{1}{\mu}C_0 - bM) = \frac{\tau}{\mu^2} - 2b,$$

which implies that

$$b = \frac{1}{2}(\frac{\tau}{\mu^2} + \gamma).$$

Therefore

$$e = \frac{-1}{\mu}C_0 + \frac{1}{2}(\frac{\tau}{\mu^2} + \gamma)M.$$

Thus

$$\begin{aligned} F.C_0 &= e.C_0 = [\frac{-1}{\mu}C_0 + \frac{1}{2}(\frac{\tau}{\mu^2} + \gamma)M]C_0 \\ &= \frac{-\tau}{\mu} + \frac{1}{2}(\frac{\tau}{\mu} + \gamma\mu) \\ &= \frac{1}{2}(-\frac{\tau}{\mu} + \gamma\mu). \end{aligned}$$

b) From the formula for e in the proof of a) it is not difficult to arrive at the proof of b).

c) Since $C_0.M = \mu$, it follows that $\pi_*(C_0) = \mu C$. Then from b) we obtain c). □

We end this subsection commenting on conditions 2) and 3) of Theorem 2. By Proposition 2.8 in Chapter 5 of [20], there is a line bundle $\mathcal{M} \to C$ so that the vector bundle $\mathcal{E}' = \mathcal{E} \otimes \mathcal{M}$ is normalized in the following sense: $H^0(\mathcal{E}') \neq 0$ but for all line bundle $\mathcal{L} \to C$ with $c_1(\mathcal{L}) < 0$ then $H^0(\mathcal{E}' \otimes \mathcal{L}) = 0$. A canonical section

$C_0 \subset F$ can be associated to such a normalized $\mathcal{E}'$. The intersection between C_0 and M is 1. Moreover, the number

$$\tau_0 = C_0.C_0 = c_1(\mathcal{E}') = c_1(\mathcal{E}) + 2c_1(\mathcal{M}),$$

is an invariant of F.

Condition 3) of Theorem 2 implies the existence of an effective curve $V \subset F$ for which $V.V < 0$ and $V.M > 0$. We now show that such an effective curve exists if and only if the invariant τ_0 is < 0. In condition 2) of Theorem 2, the requirement that C is not the only effective curve in its cohomology class is not needed if there exists a non-zero effective curve $V \subset F$ so that $F.V \geq 0$. We now also show that if $\gamma < 0$ and $\tau_0 \geq 0$ then such a curve V does not exist.

Lemma 5. *Assume that the invariant τ_0 of F is non-negative. Then*

a) For any effective curve $V \subset F$ we have $V.V \geq 0$.

b) If moreover $\gamma < 0$ then for any non-zero effective curve $V \subset F$ we have $F.V < 0$.

Proof. a) It suffices to prove for the case V is an irreducible curve. Numerically, we write $V = aC_0 + bM$. If $V = C_0$ then $V.V = \tau_0 \geq 0$. If $V = M$ then $V.V = 0$. Hence we may assume that $V \neq C_0, M$.

We consider two cases:

Case 1: $\tau_0 = 0$. By Proposition 2.20 in Chapter 5 in [20], we have $a > 0$ and $b \geq 0$. Therefore

$$V.V = a^2\tau_0 + 2ab \geq 0.$$

Case 2: $\tau_0 > 0$. By Proposition 2.21 in Chapter 5 in [20], there are two subcases:

Subcase 2.1: $a = 1, b \geq 0$. Then

$$V.V = \tau_0 + 2b \geq 0.$$

Subcase 2.2: $a \geq 2, b \geq -a\tau_0/2$. Then

$$V.V = a^2\tau_0 + 2ab \geq a^2\tau_0 + 2a(-a\tau_0/2) = 0.$$

b) It suffices to prove for the case V is an irreducible curve. If $V = M$ then $F.M = -1 < 0$. If $V = C_0$ then by Lemma 4 with $\tau = \tau_0 \geq 0$ and $\mu = 1$

$$F.C_0 = \frac{1}{2}(\gamma - \tau_0) \leq \frac{1}{2}\gamma < 0$$

because $\gamma < 0$. Therefore we may assume that $V \neq C_0, M$, and then proceed as in the proof of a). $\square$

2.4. **Dynamical degrees and entropy.** Let $f : X \to X$ be a surjective holomorphic map of a compact Kähler manifold of dimension k. For $1 \leq p \leq k$, we define the p-th dynamical degree $\lambda_p(f)$ of f to be the spectral radius of the linear map $f^* : H^{p,p}(X) \to H^{p,p}(X)$. The dynamical degrees are all ≥ 1, and are log-concave, i.e. $\lambda_j(f)^2 \geq \lambda_{j+1}(f)\lambda_{j-1}(f)$ (see Dinh-Sibony [16][15]).

Let d be a metric on X. A subset E of X is called (n, ϵ)-separated if for any pair $x \neq y \in E$ then $\max\{d(f^i(x), f^i(y)) : \ 0 \leq i \leq n-1\} \geq \epsilon$. Denote by $N(n, \epsilon)$ the maximal cardinality of an (n, ϵ)-separated set. Then the topological entropy of f is given by

$$h_{top}(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon).$$

Gromov [18] and Yomdin [32] proved the following result, relating dynamical degrees to topological entropy:

Theorem 4. *Assumptions as above. Then* $h_{top}(f) = \max_{1\leq p\leq k}\log \lambda_p(f)$.

Apply the log concavity of dynamical degrees to Gromov-Yomdin's theorem, we deduce that $h_{top}(f) = 0$ if and only if f is an automorphism and $\lambda_p(f) = 1$ for some (and hence, all) $1 \leq p \leq k-1$.

3. Proofs of Theorems 1 and 2

For the proof of Theorems 1 and 2, we first introduce the following set of cohomology classes, which uses a weaker notion of positivity than that of nef classes.

Definition 5. *Let X be a projective manifold of dimension 3. We define by $\mathcal{B}(X)$ the set of cohomology classes $\eta \in H^{1,1}(X)$ satisfying the following conditions:*

1) η is psef.

2) For every effective curve V in X then $\eta.V \geq 0$.

We also introduce a larger set of cohomology classes

Definition 6. *Let X be a projective manifold of dimension 3. We define by $\mathcal{C}(X)$ the set of cohomology classes $\eta \in H^{1,1}(X)$ satisfying the following conditions:*

1) η is psef.

2) There is a finite number of irreducible curves $V_1, \ldots, V_t \subset X$ (these curves depend on η) so that if V is an irreducible curve in X with $\eta.V < 0$, then V is one of the curves $V_1, \ldots, V_t$.

We have obvious inclusions $H^{1,1}_{nef}(X) \subset \mathcal{B}(X) \subset \mathcal{C}(X)$. The following properties of $\mathcal{B}(X)$ and $\mathcal{C}(X)$ make them useful in induction arguments involving finite blowups in dimension 3.

Lemma 6. *Let $\pi : X \to Y$ be a blow up of a projective 3-manifold Y along a point $p \in Y$ or a smooth curve $C \subset Y$.*

a) If η is in $\mathcal{C}(X)$ then $\pi_*(\eta)$ is in $\mathcal{C}(Y)$.

b) If π is a point blowup and $\eta \in \mathcal{B}(X)$ then $\pi_*(\eta) \in \mathcal{B}(Y)$.

c) If π be a blowup of Y along a smooth curve $C \subset Y$ so that C is not the only effective curve in its cohomology class, then $\pi_*(\mathcal{B}(X)) \subset \mathcal{B}(Y)$.

Proof. b) Consider the case when $\pi : X \to Y$ is a point blowup. Let $E = \mathbb{P}^2$ be the exceptional divisor and let $L \subset E$ be a line. Let $\eta \in \mathcal{B}(X)$ and let $\xi = \pi_*(\eta)$. Then ξ is psef since η is so. Let $V \subset Y$ be an irreducible curve, and let $\widetilde{V} \subset X$ be the strict transform of V. Then $\pi_*(\widetilde{V}) = V$, and $\widetilde{V}$ is not contained in $\widetilde{E}$ therefore $\widetilde{V}.E \geq 0$.

We can write $\eta = \pi^*(\xi) - \alpha E$ where $\alpha = \eta.L \geq 0$ since $\eta \in \mathcal{B}(X)$. Hence

$$\xi.V = \xi.\pi_*(\widetilde{V}) = \pi^*(\xi).\widetilde{V} = (\eta + \alpha E).\widetilde{V} \geq 0.$$

Thus $\xi \in \mathcal{B}(X)$.

a) We consider first the case when $\pi : X \to Y$ is a blowup of Y along a smooth curve $C \subset Y$. Let F be the exceptional divisor of the blowup. Let M be a fiber of $F \to C$. Let $\eta \in \mathcal{C}(X)$ and let $\xi = \pi_*(\eta)$. Then $\eta = \pi^*(\xi) - \alpha F$, where $\alpha = \eta.M$. Observe that $\alpha \geq 0$, because η can have negative intersections with only a finite number of irreducible curves while we have infinitely many fibers.

Since η is psef, it follows that ξ is psef as well. Let $V \subset Y$ be an irreducible curve which is not contained in the union of C with the images of the irreducible curves having negative intersections with η. Then we can proceed as in the proof of b) to show that $\xi.V \geq 0$. Hence $\xi \in \mathcal{C}(X)$.

The proof of the case π is a point blowup is similar.

c) Let $\pi : X \rightarrow Y$ is a blowup of Y along a smooth curve $C \subset Y$, where C is not the only effective curve in its cohomology class. Let F be the exceptional divisor of the blowup. Let M be a fiber of $F \rightarrow C$. Let $\eta \in \mathcal{C}(X)$ and let $\xi = \pi_*(\eta)$. Then $\eta = \pi^*(\xi) - \alpha F$, where $\alpha = \eta.M \geq 0$.

If $V \subset Y$ is an irreducible curve different from C, then by using its strict transform in X we can show as in the proof of b) that $\xi.V \geq 0$. If $V = C$, then since C is not the only effective curve in its cohomology class, we can find an effective curve C' having the same cohomology class as that of C so that the support of C' does not contain C. Then we can proceed as in the first case. □

Lemma 7. *Let $\pi : X \rightarrow Y$ be a finite composition $X = X_n \rightarrow X_{n-1} \rightarrow \ldots \rightarrow X_1 \rightarrow Y$ of point or curve blowups. Assume that the first map $X_1 \rightarrow Y$ is a point blowup. Assume moreover that the images of the exceptional divisors of the map $X \rightarrow X_1$ are contained in the exceptional divisor of $X_1 \rightarrow Y$.*

Let $\eta \in \mathcal{C}(X)$ and let $V \subset X$ be an effective curve such that $\eta.\eta = V$ and $\pi_*(V) = 0$. Then $V = 0$. If moreover $\eta \in \mathcal{B}(X)$ then the pushforward of η under the map $X \rightarrow X_j$ is in $\mathcal{B}(X_j)$.

Proof. We prove this by induction on n.

The initial case $n = 1$: Let $E_1 = \mathbb{P}^2$ be the exceptional divisor of the map $p : X_1 \rightarrow Y$. Then we need to show that if V is an effective curve with support in E_1 and $\eta \in \mathcal{C}(X_1)$ so that $\eta.\eta = V$ then $V = 0$. Let us denote $\xi = \pi_*(\eta) \in \mathcal{C}(Y)$. Let L_1 be a line in $E_1 = \mathbb{P}^2$. Then $\eta = \pi^*(\xi) - \alpha E_1$ where $\alpha = \eta.L_1 \geq 0$ (as in the proof of a) of Lemma 6). Since support of V is in E_1 and V is effective, there is a number $\beta \geq 0$ so that $V = \beta L_1$. Since $E_1.E_1 = -L_1$ and $E_1.\pi^*(\xi) = 0$, we have

$$\pi^*(\xi.\xi) - \alpha^2 L_1 = \eta.\eta = \beta L_1.$$

Since $\pi^*(\xi.\xi)$ and L_1 are linearly independent, it follows that $-\alpha^2 = \beta$. Since both α and β are non-negative, we get $\beta = 0 = \alpha$. Thus $V = 0$ as claimed. If moreover $\eta \in \mathcal{B}(X)$, from the fact that $\eta = p^*(\xi)$, it follows easily that $\xi \in \mathcal{B}(Y)$ as well.

Now assume that we had the claim for $n = j$. We will prove it for $n = j+1$. Let p denote the map $X_{j+1} \rightarrow X_j$. Let $\eta \in \mathcal{C}(X_{j+1})$ and $V \subset X_{j+1}$ an effective curve so that $\eta.\eta = V$ and $\pi_*(V) = 0$ in $H^{2,2}(Y)$. We need to show that $V = 0$.

We consider two cases:

Case 1: p is a point blowup. Let $E = \mathbb{P}^2$ be the exceptional divisor of p, and let $L \subset E$ be a line. Let $\xi = p_*(\eta) \in \mathcal{C}(X_j)$, and write $\eta = p^*(\xi) - \alpha E$ for $\alpha = \eta.L \geq 0$. Then $\eta.\eta = V$ becomes

$$p^*(\xi.\xi) - \alpha^2 L = V.$$

Push-forward this equation by p we obtain $\xi.\xi = p_*(V)$. Since the push-forward of V under the map $X_{j+1} \rightarrow Y$ is zero, it follows that the push-forward of $p_*(V)$ under the map $X_j \rightarrow Y$ is zero. Therefore the induction assumption implies that $p_*(V) = 0$. Therefore V must be a multiple of L, and we can write $V = \beta L$ for some $\beta \geq 0$. Also $\xi.\xi = 0$ and thus $p^*(\xi.\xi) = 0$. Replace this into the original equation we get $-\alpha^2 L = \beta L$ which implies $\beta = \alpha = 0$, i.e. $V = 0$. If moreover $\eta \in \mathcal{B}(X_{j+1})$, from the fact that $\eta = p^*(\xi)$, it follows easily that $\xi \in \mathcal{B}(X_j)$ as well.

Case 2: p is a blowup of a smooth curve $C \subset X_j$ so that the push-forward of C under the map $X_j \to Y$ is 0. Let F be the exceptional divisor of p and let $M \subset F$ be a fiber of the map $F \to C$. Let $\xi = p_*(\eta) \in \mathcal{C}(X_j)$, and write $\eta = p^*(\xi) - \alpha F$ for $\alpha = \eta.M \geq 0$. Then $\eta.\eta = V$ becomes

$$p^*(\xi.\xi) - 2\alpha\pi^*(\xi).F + \alpha^2(F.F) = V.$$

Push-forward this equation by p we obtain $\xi.\xi = \alpha^2 C + p_*(V)$. Since the push-forward of V under the map $X_{j+1} \to Y$ is zero, it follows that the push-forward of $p_*(V)$ under the map $X_j \to Y$ is zero. Therefore, the class $\alpha^2 C + p_*(V)$ is effective and has image zero under push-forward by the map $X_j \to Y$. Apply the induction assumption we have that $\xi.\xi = 0 = \alpha^2 C + p_*(V)$ and hence $\alpha = 0$. The original equation becomes $0 = V$, and we are done. If moreover $\eta \in \mathcal{B}(X_{j+1})$, from the fact that $\eta = p^*(\xi)$ and the existence of a section $C_0 \subset F$ (see Section 2.3), we have $\xi.C = \eta.C_0 \geq 0$ and it easily follows that $\xi \in \mathcal{B}(Y)$ as well. □

Now we prove a general result on non-existence of automorphisms of positive entropies (see also Lemma 2.4 and other results in Zhang [34], and Dinh-Sibony [17]).

Theorem 7. *Let X be a projective manifold of dimension 3 and let $f : X \to X$ be an automorphism. Assume that whenever $\eta \in H^{1,1}_{nef}(X)$ is an eigenvector of $f^* : H^{1,1}(X) \to H^{1,1}(X)$ then either $\eta.\eta \neq 0$ or $\eta \in \mathbb{R}.H^{1,1}(X, \mathbb{Q})$, i.e. there is a real number a and a class $\eta_0 \in H^{1,1}(X, \mathbb{Q})$ so that $\eta = a\eta_0$ (in other words, η is proportional to a rational cohomology class and hence to an integral class). Then $\lambda_1(f) = \lambda_2(f) = 1$, and therefore $h_{top}(f) = 0$.*

Proof. Assume in contrast that $\lambda_1(f) > 1$. Since f^* preserves the cone $H^{1,1}_{nef}(X)$, by a Perron-Frobenius type theorem, there is a non-zero nef class η so that $f^*(\eta) = \lambda_1(f)\eta$.

First we claim that for such an η, then $\eta.\eta \neq 0$. Otherwise, by assumption we can write $\eta = a\eta_0$ for some real number $a \in \mathbb{R}$ and $\eta_0 \in H^{1,1}(X, \mathbb{Q})$. Dividing by a we may assume that $\eta = \eta_0$ is in $H^{1,1}(X, \mathbb{Q})$. Since f^* preserves $H^{1,1}(X, \mathbb{Q})$, it follows from $f^*(\eta) = \lambda_1(f)\eta$ that $\lambda_1(f) \in \mathbb{Q}$. However, the latter is irrational (see e.g. Zhang [34] and Bedford [3]). [For the convenience of the readers, we reproduce the proof of this fact here. Let A be the matrix of $f^* : H^2(X, \mathbb{C}) \to H^2(X, \mathbb{C})$, then A is an integer matrix, and $\lambda_1(f)$ is a real eigenvalue of A. Moreover, A is invertible and its inverse A^{-1} is the matrix of the map $(f^{-1})^* : H^2(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ hence is also an integer matrix. Therefore $det(A) = \pm 1$. Thus the characteristic polynomial $P(x)$ of A is a monic polynomial of integer coefficients and $P(0) = \pm 1$. Assume that $\lambda_1(f)$ is a rational number. Since $\lambda_1(f)$ is an algebraic integer, it follows that $\lambda_1(f)$ must be an integer. Then we can write $P(x) = (x - \lambda_1(f))Q(x)$, here $Q(x)$ is a polynomial of integer coefficients. If $\lambda_1(f) > 1$ we get a contradiction $\pm 1 = P(0) = -\lambda_1(f)Q(0)$]. Thus $\eta.\eta \neq 0$ as claimed.

Therefore $\eta.\eta$ is an eigenvector of $f^* : H^{2,2}(X) \to H^{2,2}(X)$ of eigenvalue $\lambda_1(f)^2$. Hence $\lambda_2(f) \geq \lambda_1(f)^2$. Since eigenvectors of $f^* : H^{1,1}(X) \to H^{1,1}(X)$ and $(f^{-1})^* : H^{1,1}(X) \to (f^{-1})^* : H^{1,1}(X) \to H^{1,1}(X)$ are the same (the operators f^* and $(f^{-1})^*$ are inverse to each other), we can apply the same argument to the inverse f^{-1} to obtain $\lambda_2(f^{-1}) \geq \lambda_1(f^{-1})^2$. But $\lambda_1(f^{-1}) = \lambda_2(f)$ and $\lambda_2(f^{-1}) = \lambda_1(f)$, since $(f^{-1})^* = f_*$ is conjugate to f^*. In fact, let ω_X be a Kähler $(1,1)$ form on X. Then (see Dinh-Sibony [15][16])

$$\begin{aligned}\lambda_1(f^{-1}) &= \lim_{j\to\infty}(\int_X (f^{-j})^*(\omega_X)\wedge\omega_X^2)^{1/j} = \lim_{j\to\infty}(\int_X (f^j)_*(\omega_X)\wedge\omega_X^2)^{1/j}\\ &= \lim_{j\to\infty}(\int_X \omega_X\wedge (f^j)^*(\omega_X^2))^{1/j} = \lambda_2(f),\end{aligned}$$

and similarly for the equality $\lambda_2(f^{-1}) = \lambda_1(f)$.

Hence we must have $\lambda_1(f) = \lambda_1(f)^2 = \lambda_2(f) = \lambda_2(f)^2 = 1$, as claimed. □

Now we are ready to give the proofs of Theorems 2 and 1.

Proof. (Of Theorem 2) Let $X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathbb{P}^3$ be as in the statement of Theorem 2. To prove Theorem 2, it suffices to show that X satisfies the conditions of Theorem 7. Indeed we will prove a stronger condition:

Condition (A): If $\eta \in \mathcal{B}(X)$ satisfies $\eta.\eta = 0$ then $\eta \in \mathbb{R}.H^{1,1}(X,\mathbb{Q})$.

We prove this by induction on n.

a) The initial step $n = 0$ is clear, since then $X = X_0 = \mathbb{P}^3$ and hence if $\eta \in H^{1,1}(X)$ is such that $\eta.\eta = 0$ then $\eta = 0$.

b) We show that if $\pi : X = X_1 \to \mathbb{P}^3$ is the blowup of points $p_1,\ldots,p_t \in \mathbb{P}^3$ and smooth curves $C_1,\ldots,C_s \subset \mathbb{P}^3$ in general positions, then X satisfies Condition (A). Let $H \subset \mathbb{P}^3$ be the class of a generic hyperplane. Let $E_1,\ldots,E_t$ be the exceptional divisors corresponding with $e_1,\ldots,e_t$, and let $L_i \subset E_i$ be a line. Let $F_1,\ldots,F_s$ be the exceptional divisors corresponding to $C_1,\ldots,C_s$, and let $M_j \subset F_j$ be a fiber of the projection $F_j \to C_j$. Let $d_j \geq 1$ be the degree of C_j (hence $d_j = H.C_j$ in $\mathbb{P}^3$), and let $g_j \geq 0$ be the genus of C_j.

The cohomology group $H^{1,1}(X)$ is generated by $H, E_1,\ldots,E_t,F_1,\ldots,F_s$, and the cohomology group $H^{2,2}(X)$ is generated by $H.H, L_1,\ldots,L_t,M_1,\ldots,M_s$. The intersection product on X is as follows (see, e.g., Section 6 Chapter 4 in the book of Griffiths and Harris [19])

$$\begin{aligned}&H.E_i = 0,\ H.F_i = d_iM_i,\\ &E_i.E_j = -\delta_{i,j}L_i,\ E_i.F_j = 0.\\ &F_i.F_j = \delta_{i,j}[-d_iH.H + (4d_i + 2g_i - 2)M_i].\end{aligned}$$

If $\eta \in H^{1,1}(X,\mathbb{R})$ we can write $\eta = aH - \sum_i e_iE_i - \sum_j f_jF_j$ for real numbers $a, e_1,\ldots,e_t,f_1,\ldots,f_s$. Then a computation shows

$$\begin{aligned}&\eta.\eta\\ =\ & a^2H.H - 2aH(\sum_i e_iE_i) - 2aH(\sum_j f_jF_j) + (\sum_i e_iE_i)^2 + (\sum_j f_jF_j)^2 + 2(\sum_i e_iE_i).(\sum_j f_jF_j)\\ =\ & a^2H.H - 2a\sum_j d_jf_jM_j - \sum e_i^2L_i + \sum_j f_j^2[-d_jH^2 + (4d_j + 2g_j - 2)M_j].\end{aligned}$$

Therefore $\eta.\eta = 0$ if and only if

$$\begin{aligned}a^2 &= \sum_j d_jf_j^2,\\ e_i^2 &= 0,\ \forall i = 1,\ldots,t,\\ 2ad_jf_j &= f_j^2(4d_j + 2g_j - 2),\ \forall j = 1,\ldots,s.\end{aligned}$$

From the equations for e_i we have that $e_i = 0$. If $a = 0$ then the first equation $a^2 = \sum_j d_j f_j^2$ implies that $a = f_1 = \ldots = f_s = 0$ as well, and hence $\eta = 0$. Assume now $a \neq 0$. If $f_j \neq 0$ then from the equation for f_j we have

$$\frac{f_j}{a} = \frac{2d_j}{4d_j + 2g_j - 2} \in \mathbb{Q},$$

and therefore

$$\eta = a(H - \sum_j \frac{f_j}{a} F_j) \in \mathbb{R}H^{1,1}(X, \mathbb{Q}),$$

as wanted.

c) Let $X \to Y$ be a finite composition of blowups along smooth centers, the images in Y of whose exceptional divisors are points. We now show that if Y satisfies the assumptions of Condition (A), then X does also. Without loss of generality, we may assume that $X \to Y$ can be decomposed as $X = X_n \to X_{n-1} \to \ldots \to X_1 \to Y$, where $X_1 \to Y$ is a point blowup and the images of the exceptional divisors of $X \to Y$ is that point.

We need to show that if $\eta \in \mathcal{B}(X)$ be such that $\eta.\eta = 0$ then $\eta \in \mathbb{R}H^{1,1}(X, \mathbb{Q})$. We prove this by induction on n.

Initial case $n = 1$: $X \to Y = \pi : X_1 \to Y$ be blowup at one point. Let E be the exceptional divisor and let L be a line in E. Let $\eta \in \mathcal{B}(X)$ be so that $\eta.\eta = 0$. Let $\xi = \pi_*(\eta)$. Then by Lemma 6, $\xi \in \mathcal{B}(Y)$. We can write $\eta = \pi^*(\xi) - \alpha E$ for some constant $\alpha \geq 0$. Computing as in Section 2 we obtain

$$0 = \eta.\eta = \pi^*(\xi.\xi) - \alpha^2 L.$$

Intersecting the RHS of the above equality with E, it follows that $\alpha = 0$ and therefore $\pi^*(\xi.\xi) = 0$ as well. Hence $\xi.\xi = 0$, and by the induction assumption, it follows that $\xi \in \mathbb{R}H^{1,1}(X, \mathbb{Q})$. Consequently, $\eta = \pi^*(\xi) \in \mathbb{R}H^{1,1}(X, \mathbb{Q})$.

Assume by induction that the claim is true for $n = j$. We prove that it is true for $n = j + 1$.

We consider two cases:

Case 1: $p : X_{j+1} \to X_j$ is a point blowup. Let $E = \mathbb{P}^2$ be the exceptional divisor and let $L \subset E$ be a line. Let $\eta \in \mathcal{B}(X_{j+1})$ be such that $\eta.\eta = 0$. We need to show that $\eta \in \mathbb{R}H^{1,1}(X_{j+1}, \mathbb{Q})$. Let us write $\xi = p_*(\eta) \in \mathcal{B}(X_j)$ and $\eta = p^*(\xi) - \alpha E$ for $\alpha \geq 0$. Then $\eta.\eta = 0$ becomes $p^*(\xi.\xi) - \alpha^2 L = 0$ and hence $p^*(\xi.\xi) = \alpha^2 L = 0$ since they are linearly independent. Thus $\alpha = 0$ and $\xi.\xi = 0$. Apply induction assumption we get $\xi \in \mathbb{R}H^{1,1}(X_j, \mathbb{Q})$ and therefore $\eta = p^*(\xi) \in \mathbb{R}H^{1,1}(X_{j+1}, \mathbb{Q})$.

Case 2: $p : X_{j+1} \to X_j$ is a blowup at a smooth curve $C \subset X_j$ so that the push-forward of C under the map $X_j \to Y$ is zero. Let F be the exceptional divisor of the map p, and let $M \subset F$ be a fiber of the projection $F \to C$. Let $\eta \in \mathcal{B}(X_{j+1})$ be such that $\eta.\eta = 0$. We need to show that $\eta \in \mathbb{R}H^{1,1}(X_{j+1}, \mathbb{Q})$. Let us write $\xi = p_*(\eta)$ which is in $\in \mathcal{C}(X_j)$ by Lemma 6, and $\eta = p^*(\xi) - \alpha F$ for $\alpha \geq 0$. Then $\eta.\eta = 0$ becomes $p^*(\xi.\xi) - 2\alpha p^*(\xi).E + \alpha^2 E.E = 0$. Push-forward this equation by p we get $\xi.\xi = \alpha^2 C$. Apply Lemma 7, it follows that $\xi.\xi = \alpha^2 C = 0$. Hence $\alpha = 0$, $\xi \in \mathcal{B}(X_j)$ and $\eta = p^*(\xi)$. Apply induction assumption for $\xi.\xi = 0$ we get $\xi \in \mathbb{R}H^{1,1}(X_j, \mathbb{Q})$ and therefore $\eta = p^*(\xi) \in \mathbb{R}H^{1,1}(X_{j+1}, \mathbb{Q})$.

d) Let $\pi : X \to Y$ be the blowup of Y along a smooth curve $C \subset Y$ so that $\gamma := c_1(Y).C + 2g - 2 < 0$, and C is not the only effective curve in its cohomology class. We now show that if Y satisfies Condition (A), then X does so. Let F be the exceptional divisor of the blowup and let $M \subset F$ be a fiber of the projection $F \to C$.

Let $\eta \in \mathcal{B}(X)$, then $\xi = \pi_*(\eta) \in \mathcal{B}(Y)$ by Lemma 6 and the assumption on C, and there is $a \geq 0$ so that $\eta = \pi^*(\xi) - aF$. Assume that $\eta.\eta = 0$. Then

$$0 \quad = \quad \eta.\eta.F = (\pi^*(\xi.\xi) - 2a\xi.F + a^2F.F).F = 2a\xi.C - a^2\gamma.$$

Here we used that $F.F.F = -\gamma$ and $\pi_*(F.F) = -C$. From this, it follows that $a = 0$. Otherwise we can divide by $a > 0$ and obtain $2\xi.C = a\gamma$ which is a contradiction since $\xi.C \geq 0$ (because $\xi \in \mathcal{B}(Y)$) and $\gamma < 0$. Knowing $a = 0$ we can argue as at the end of the proof of c).

e) Let $\pi : X \to Y$ be the blowup of Y along a smooth curve $C \subset Y$ so that there is an irreducible hypersurface $S \subset Y$ containing C satisfying condition 3) of Theorem 2. As the last step of the proof of Theorem 2, we now show that if Y satisfies Condition (A), then X does so. Let F be the exceptional divisor of the blowup and let $M \subset F$ be a fiber of the projection $F \to C$.

Let $\eta \in \mathcal{B}(X)$, then $\xi = \pi_*(\eta) \in \mathcal{C}(Y)$ by Lemma 6, and there is $a \geq 0$ so that $\eta = \pi^*(\xi) - aF$. Assume that $\eta.\eta = 0$. Then

$$0 \quad = \quad \eta.\eta.F = (\pi^*(\xi.\xi) - 2a\xi.F + a^2F.F).F = 2a\xi.C - a^2\gamma.$$

Here we used that $F.F.F = -\gamma$ and $\pi_*(F.F) = -C$. From this, it follows that $a = 0$. Otherwise we can divide by $a > 0$ and obtain $2\xi.C = a\gamma$. We now construct an effective curve $C_0 \subset F$ and use it to derive a contradiction.

Recall that $\kappa = S.C$, and $\mu \geq 1$ is the multiplicity of C in S. Then the strict transform $\widetilde{S}$ of S is given by $\widetilde{S} = \pi^*(S) - \mu F$, and is an irreducible hypersurface of X. Since $\widetilde{S}$ and F are different irreducible hypersurfaces, their intersection $C_0 = \widetilde{S}.F = (\pi^*(S) - \mu F).F$ is an effective curve of F. We now compute the numbers $C_0.C_0$ and $C_0.M$. We have

$$\begin{aligned} C_0.C_0 &= \widetilde{S}|_F.\widetilde{S}|_F = \widetilde{S}.\widetilde{S}.F \\ &= (\pi^*(S) - \mu F).(\pi^*(S) - \mu F).F = -2\mu\pi^*(S).F.F + \mu^2F.F.F \\ &= 2\mu S.C - \mu^2\gamma = 2\mu\kappa - \mu^2\gamma. \end{aligned}$$

Denote by $\tau = C_0.C_0$ and $\mu_0 = C_0.M$. Note that $\mu_0 \neq 0$, otherwise we have C_0 is a multiplicity of M, and hence $\pi_*(C_0) = 0$. But from the definition of C_0 we can see that $\pi_*(C_0) = \mu C \neq 0$. Then by the computations at the end of Section 2, we have

$$F.F = -\frac{1}{\mu_0}C_0 + \frac{1}{2}(\frac{\tau}{\mu_0^2} + \gamma)M.$$

Pushforward this by the map π, using that $\pi_*(F.F) = -C$ and $\pi_*(C_0) = \mu C$ we have that $\mu_0 = \mu$.

By Lemma 4 and the above computation $\tau = 2\mu\kappa - \mu^2\gamma$, we obtain

$$F.C_0 = \frac{1}{2}(\gamma\mu - \frac{\tau}{\mu}) = \gamma\mu - \kappa.$$

Because $\eta \in \mathcal{B}(X)$ and $2\xi.C = a\gamma$, it follows that

$$\begin{aligned} 0 &\leq \eta.C_0 = (\pi^*(\xi) - aF).C_0 = \mu\xi.C - \frac{a}{2}(\gamma\mu - \frac{\tau}{\mu}), \\ &= \frac{a}{2}\gamma\mu - \frac{a}{2}(\gamma\mu - \frac{\tau}{\mu}) = \frac{a}{2}\frac{\tau}{\mu} = a(\kappa - \frac{1}{2}\gamma\mu). \end{aligned}$$

This contradicts the assumptions that $2\kappa < \gamma\mu$ and $a > 0$. Therefore $a = 0$. Knowing that $a = 0$, it follows that $\xi \in \mathcal{B}(Y)$ and we can use the induction assumption for it to have $\xi \in \mathbb{R}.H^{1,1}(Y,\mathbb{Q})$ and therefore $\eta = \pi^*(\xi) \in \mathbb{R}.H^{1,1}(X,\mathbb{Q})$. □

Proof. (Of Theorem 1)

Let $\pi : X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathbb{P}^3$ be a finite composition of blowups along smooth centers satisfying conditions of Theorem 1.

We first show the following:

1) Claim 1: If $\eta \in \mathcal{C}(X)$ and $\eta \neq 0$, then either $\eta.\eta \neq 0$ or $K_X.\eta \neq 0$. Here K_X is the canonical divisor of X.

Proof (of Claim 1): We prove the claim by induction on n.

The initial case $n = 0$: Then $X = \mathbb{P}^3$, $H^{1,1}(X)$ is generated by a generic hyperplane $H \subset \mathbb{P}^3$, $K_X = -4H$ and $\eta = aH$ for some $a > 0$. Hence both $\eta.\eta$ and $K_X.\eta$ are non-zero.

Assume that the claim is true for $n = j$. We now show that it is true for $n = j+1$. We define by p the map $X_{j+1} \to X_j$. Let $\eta \in \mathcal{C}(X_{j+1})$, which is non-zero, we now show that at least one of the expressions $\eta.\eta$ and $K_{X_{j+1}}.\eta \neq 0$.

We define $\xi = p_*(\eta) \in \mathcal{C}(X_j)$. We consider two cases:

Case 1: $p : X = X_{j+1} \to X_j = Y$ is a point blowup. Let $E = \mathbb{P}^2$ be the exceptional divisor of p and let $L \subset E$ be a line. Then $K_X = p^*(K_Y) + 2E$. We can write $\eta = p^*(\xi) - \alpha E$ for $\alpha = \eta.L \geq 0$.

If we had both $\eta.\eta = 0$ and $K_X.\eta = 0$ then we have

$$\begin{aligned} 0 &= \eta.\eta = (p^*(\xi) - \alpha E).(p^*(\xi) - \alpha E) = p^*(\xi.\xi) - \alpha^2 L, \\ 0 &= K_X.\eta = (p^*(K_Y) + 2E).(p^*(\xi) - \alpha E) = p^*(K_Y.\xi) + 2\alpha L. \end{aligned}$$

Since $p^*(\xi.\xi)$ and L are linearly independent, from the first equation we imply that $\xi.\xi = 0$ and $\alpha = 0$. Similarly, from the second equation we have $K_Y.\xi = 0$. If $\xi \neq 0$, then by induction assumption, not both $\xi.\xi$ and $K_Y.\xi$ are zero, and we arrive at a contradiction. If $\xi = 0$, then $\eta = p^*(\xi) = 0$ as well, and we have a contradiction again. Therefore Claim 1 is proved in Case 1.

Case 2: $p : X = X_{j+1} \to X_j = Y$ is a blowup of Y along a smooth curve $C \subset Y$ for which $c_1(Y).C \neq 2g-2$, where $c_1(Y) = -K_Y$ is the first Chern class of Y and g is the genus of C. Let F be the exceptional divisor of p, and let $M \subset F$ be a fiber of the projection $F \to C$. Let g be the genus of C, and let $c_1(Y) = -K_Y$ be the first Chern class of Y. Then $K_X = p^*(K_Y) + F$. We can write $\eta = p^*(\xi) - \alpha F$ for $\alpha = \eta.L \geq 0$.

We define

$$\gamma = c_1(Y).C + 2g - 2.$$

If we had both $\eta.\eta = 0$ and $K_X.\eta = 0$ then we have

$$\begin{aligned} 0 &= \eta.\eta = (p^*(\xi) - \alpha F).(p^*(\xi) - \alpha F) = p^*(\xi.\xi) - 2\alpha p^*(\xi).F + \alpha^2 F.F, \\ 0 &= K_X.\eta = (p^*(K_Y) + F).(p^*(\xi) - \alpha F) = p^*(K_Y.\xi) - \alpha p^*(K_Y).F + p^*(\xi).F - \alpha F.F. \end{aligned}$$

Intersecting both of these equations with F, using $F.F.F = -\gamma$ and $p_*(F.F) = -C$, we obtain

$$\begin{aligned} &2\alpha\xi.C - \alpha^2\gamma = 0, \\ &\alpha K_Y.C - \xi.C + \alpha\gamma = 0. \end{aligned}$$

Then we must have $\alpha = 0$. Otherwise, dividing 2α from the first equation we have that $\xi.C = \alpha\gamma/2$. Substituting this into the second equation and dividing by α we get $2K_Y.C = -\gamma$. Hence $c_1(Y).C = 2g - 2$, which is a contradiction.

Now that we have $\alpha = 0$, the original equations become $p^*(\xi.\xi) = 0$ and $p^*(K_Y.\xi) + p^*(\xi).E = 0$. Push-forward both of these equations to Y, we obtain that $\xi.\xi = 0$ and $K_Y.\xi = 0$ and can proceed as in Case 1.

2) Now we continue with the proof of Theorem 1. Let $f \in Aut(X)$.

We first show that $\lambda_2(f) \geq \lambda_1(f)$. To this end, let η be a non-zero nef class which is an eigenvector of eigenvalue $\lambda_1(f) \geq 1$ of $f^* : H^{1,1}(X) \to H^{1,1}(X)$. If $\eta.\eta \neq 0$, then $\eta.\eta$ is an eigenvector of eigenvalue $\lambda_1(f)^2$ of $f^* : H^{2,2}(X) \to H^{2,2}(X)$, therefore $\lambda_2(f) \geq \lambda_1(f)^2 \geq \lambda_1(f)$ as claimed. Otherwise, by Claim 1 we must have $K_X.\eta \neq 0$. Since f is an automorphism of X, $f^*(K_X) = K_X$. Therefore $K_X.\eta$ is an eigenvector of eigenvalue $\lambda_1(f)$ of $f^* : H^{2,2}(X) \to H^{2,2}(X)$, and we again have $\lambda_2(f) \geq \lambda_1(f)$.

If we apply the above argument to f^{-1}, we obtain $\lambda_1(f) = \lambda_2(f^{-1}) \geq \lambda_1(f^{-1}) = \lambda_2(f)$. Therefore $\lambda_1(f) = \lambda_2(f)$, and we are done. □

4. Examples

4.1. **The case $X_0 = \mathbb{P}^2 \times \mathbb{P}^1$.** By Künneth's formula, $H^{1,1}(X_0)$ is generated by the classes of $\mathbb{P}^2 \times \{pt\}$ and $\mathbb{P}^1 \times \mathbb{P}^1$ (here $\{pt\}$ means a point). The intersection on $H^{1,1}(X_0)$ is

$$\begin{aligned} \mathbb{P}^2 \times \{pt\}.\mathbb{P}^2 \times \{pt\} &= 0, \\ \mathbb{P}^2 \times \{pt\}.\mathbb{P}^1 \times \mathbb{P}^1 &= \mathbb{P}^1 \times \{pt\}, \\ \mathbb{P}^1 \times \mathbb{P}^1.\mathbb{P}^1 \times \mathbb{P}^1 &= \{pt\} \times \mathbb{P}^1. \end{aligned}$$

By Künneth's formula again, $H^{2,2}(X_0)$ is generated by $\mathbb{P}^1 \times \{pt\}$ and $\{pt\} \times \mathbb{P}^1$. The pairing between $H^{1,1}(X_0)$ and $H^{2,2}(X_0)$ is given by

$$\begin{aligned} \mathbb{P}^2 \times \{pt\}.\mathbb{P}^1 \times \{pt\} &= 0, \\ \mathbb{P}^2 \times \{pt\}.\{pt\} \times \mathbb{P}^1 &= 1, \\ \mathbb{P}^1 \times \mathbb{P}^1.\mathbb{P}^1 \times \{pt\} &= 1, \\ \mathbb{P}^1 \times \mathbb{P}^1.\{pt\} \times \mathbb{P}^1 &= 0. \end{aligned}$$

a) We first check that the space $X_0 =$ satisfies Condition (A) in the proof of Theorem 2. Let η be in $H^{1,1}_{nef}(X_0)$ so that $\eta.\eta = 0$. We need to show that $\eta \in \mathbb{R}.H^{1,1}(X_0, \mathbb{Q})$. In fact, we can write $\eta = a\mathbb{P}^2 \times \{pt\} + b\mathbb{P}^1 \times \mathbb{P}^1$ for real numbers a and b. Since η is nef, we have

$$\begin{aligned} b = \eta.\mathbb{P}^1 \times \{pt\} &\geq 0, \\ a = \eta.\{pt\} \times \mathbb{P}^1 &\geq 0. \end{aligned}$$

By computation, it follows that $\eta.\eta = 2ab\mathbb{P}^1 \times \{pt\} + b^2\{pt\} \times \mathbb{P}^1$. Therefore $\eta.\eta = 0$ if and only if $ab = b^2 = 0$, i.e. $b = 0$. Hence $\eta = a\mathbb{P}^2 \times \{pt\} \in \mathbb{R}.H^{1,1}(X_0, \mathbb{Q})$, as wanted.

b) We next show the following: Let $p_1, \ldots, p_n$ and p be pairwise distinct points in $\mathbb{P}^2$. Let $\pi : X \to X_0 = \mathbb{P}^2 \times \mathbb{P}^1$ be the blowup at curves $p_1 \times \mathbb{P}^1, \ldots, p_n \times \mathbb{P}^1$, and let $C = p \times \mathbb{P}^1$. Then C does not satisfy both conditions 2) and 3) in Theorem 2. Moreover, note that some of these spaces X does have automorphisms of positive entropies (see Remarks right after Theorem 2). Therefore, we see that the conditions of Theorem 2 are somewhat optimal.

Proof. Using Whitney's sum formula for Chern classes, we find that $c_1(X_0) = 2\mathbb{P}^2 \times \{pt\} + 3\mathbb{P}^1 \times \mathbb{P}^1$. Denote by Z the blowup of $\mathbb{P}^2$ at the points $p_1, \ldots, p_n$, and let $E_1, \ldots, E_n$ be the corresponding exceptional divisors. Then X is biholomorphic equivalent to $Z \times \mathbb{P}^1$, and $c_1(X) = \pi^*(2\mathbb{P}^2 \times \{pt\} + 3\mathbb{P}^1 \times \mathbb{P}^1) - \sum_j E_j \times \mathbb{P}^1$. The curve $C = p \times \mathbb{P}^1$ has genus $g = 0$, and has intersections 0 with the exceptional divisors $E_j \times \mathbb{P}^1$ since p is different from p_j. Therefore

$$\begin{aligned}\gamma = c_1(X).C + 2g - 2 &= (2\mathbb{P}^2 \times \{pt\} + 3\mathbb{P}^1 \times \mathbb{P}^1).p \times \mathbb{P}^1 - 2\\ &= 2 - 2 = 0.\end{aligned}$$

Thus C does not satisfy condition 2) of Theorem 2.

Now let $\widetilde{S}$ be an irreducible hypersurface of X containing C. Then $\widetilde{S}$ can not be one of the exceptional divisors $E_j \times \mathbb{P}^1$, since $p \neq p_j$. Therefore, $\widetilde{S}$ must be the strict transform of a hypersurface $S \subset X_0$. Therefore, in cohomology: $\widetilde{S} = \pi^*(S) - \sum_j \mu_j E_j \times \mathbb{P}^1$, here μ_j is the multiplicity of $p_j \times \mathbb{P}^1$ in S. If we let p vary, we see that the curve C moves in a family of curves which cover the whole space $X_0 = \mathbb{P}^2 \times \mathbb{P}^1$. Hence there must be a curve in the family intersecting S at isolated points, and this shows that $S.C \geq 0$. Whatever the multiplicity μ of C in $\widetilde{S}$ is, then $\mu.\gamma = 0$. Also, C has intersections 0 with exceptional divisors $E_j \times \mathbb{P}^1$ as above. Therefore

$$\kappa = \widetilde{S}.C = S.C \geq 0 = \mu\gamma.$$

Thus condition 3) in Theorem 2 is not satisfied for C. □

From the above proof we obtain the following consequence

Corollary 2. *Let $\pi : S \to \mathbb{P}^2$ be a finite blowup of $\mathbb{P}^2$. Then for any automorphism $f : S \times \mathbb{P}^1 \to S \times \mathbb{P}^1$ we have $\lambda_1(f) = \lambda_2(f)$.*

Proof. We can see from the computation in the above proof that Theorem 1 applies: the manifold $X = S \times \mathbb{P}^1$ is obtained as a finite composition of blowups $X_{j+1} \to X_j$ along curves C_j which are isomorphic to $\mathbb{P}^1$, hence $c_1(X_j).C_j = 2 \neq 2g-2 = -2$. □

We observe that Corollary 2 is compatible with the fact that known examples of automorphisms of positive entropies of $S \times \mathbb{P}^1$ are products $g \times h : S \times \mathbb{P}^1 \to S \times \mathbb{P}^1$.

c) Finally we show the following: Let $p_1, \ldots, p_n$ and p be pairwise distinct points in $\mathbb{P}^1$. Let $D_1, \ldots, D_n$ and D be smooth curves in $\mathbb{P}^2$. Let $\pi : X \to X_0 = \mathbb{P}^2 \times \mathbb{P}^1$ be the blowup at curves $D_1 \times p_1, \ldots, D_n \times p_n$. Then the curve $C = D \times p$ does not satisfy condition 2) of Theorem 2, but it does satisfy condition 3) of Theorem 2. Therefore, any automorphism of X has topological entropy zero.

Proof. Let E_j be the exceptional divisor of the blowup corresponding to $D_j \times p_j$. Then

$$c_1(X) = \pi^*(2\mathbb{P}^2 \times \{pt\} + 3\mathbb{P}^1 \times \mathbb{P}^1) - \sum_j E_j.$$

Let $d \geq 1$ be the degree of D, and let $g \geq 0$ be its genus. Note that $C = D \times p$ is disjoint from the curves $D_j \times p_j$ since $p \neq p_j$, hence C has intersection 0 with the exceptional divisors E_j. Therefore

$$\gamma = c_1(X).C + 2g - 2 \quad = \quad (2\mathbb{P}^2 \times \{pt\} + 3\mathbb{P}^1 \times \mathbb{P}^1).D \times p = 3d + 2g - 2 \geq 1.$$

This shows that C does not satisfy condition 2) in Theorem 2.

Let $S = \mathbb{P}^2 \times p \subset \mathbb{P}^2 \times \mathbb{P}^1$, which can be identified with its strict transform in X since S has no intersection with the centers of blowups. Then S is an irreducible hypersurface in X containing $C = D \times p$, and the multiplicity of C in S is $\mu = 1$. Moreover,

$$\kappa = S.C = \mathbb{P}^2 \times p.D \times p = 0.$$

Hence $2\kappa = 0 < 1 \leq \mu\gamma$, which shows that condition 3) of Theorem 2 is satisfied. □

4.2. **The case $X_0 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.** This case is very similar to the case $X_0 = \mathbb{P}^2 \times \mathbb{P}^1$ above. The readers can easily redo all the (analogs of) computations and constructions in the previous section.

4.3. Proofs of Examples 3, 4, 5 and 6.

Proof. (Of Example 3) Let $E_1, \ldots, E_t$ be the exceptional divisors of the blowup $Y \to \mathbb{P}^3$, and let $L_1 \subset E_1, \ldots, L_t \subset E_t$ be lines. Let H be a generic hyperplane in $\mathbb{P}^3$. Let C be the strict transform of D and S is the strict transform of W. Then their classes are

$$\begin{aligned} C &= dH.H - \sum_i L_i, \\ S &= H - \sum_i E_i, \end{aligned}$$

while $c_1(Y) = 4H - 2\sum_i E_i$. Since D is a smooth plane curve, by the genus formula, the genus g of D is $g = (d-1)(d-2)/2$ which is the same as that of C. Therefore,

$$\begin{aligned} \kappa &= S.C = d - t, \\ \gamma &= c_1(Y).C + 2g - 2 = 4d - 2t + (d-1)(d-2) - 2, \\ \mu &= 1. \end{aligned}$$

Hence the inequality $2\kappa < \mu\gamma$ is the same as

$$2d + (d-1)(d-2) - 2 > 0,$$

which is satisfied when $d \geq 2$. For the case $d = 1$, the proof that X satisfies Theorem 1 when $t \neq 3$ can be done similarly to the above, see Example 5 for an explicit calculation when $t = 2$. □

Proof. (Of Example 4) The blowup $Y \to \mathbb{P}^3$ is the blowup $X_1 \to \mathbb{P}^3$ of Theorem 2, therefore satisfies Condition (A) in the proof of Theorem 2. Let F be the exceptional divisor of the blowup $Y \to \mathbb{P}^3$, and let M be a fiber of $F \to C_1$.

Let d_1 and d_2 be the degrees of C_1 and C_2. Then C_1 and C_2 intersect at $d_1.d_2$ points in W, by Bezout theorem. Let C be the strict transform of C_2 in Y, then its class is $C = d_2H.H - d_1d_2M$. Let S be the strict transform of W in Y. Then S contains C and its class is $S = H - F$. The first Chern class of Y is $c_1(Y) = 4H - F$.

We now check that C satisfies condition 3) of Theorem 2. We have $\mu = 1$,

$$\gamma = c_1(Y).C + 2g - 2 = 4d_2 - d_1d_2 + 2g - 2,$$

and

$$\kappa = S.C = d_2 - d_1d_2.$$

Therefore, since $d_1, d_2 \geq 1$ and $g \geq 0$, we have

$$\mu\gamma - 2\kappa = 2d_2 + d_1 d_2 + 2g - 2 > 0,$$

as wanted. □

Proof. (Of Example 5) Let E_0, E_1, E_2 and E_3 be the exceptional divisors of the blowup $Y \to \mathbb{P}^3$, and let $L_0 \subset E_0$, $L_1 \subset E_1$, $L_2 \subset E_2$ and $L_3 \subset E_3$ be the generic lines. Let $H \subset \mathbb{P}^3$ be a generic hyperplane. Then $C = \widetilde{\Sigma_{0,1}} = H.H - L_2 - L_3$, degree of C is $d = 1$ and its genus is $g = 0$. The first Chern class of Y is

$$c_1(Y) = 4H - 2E_0 - 2E_1 - 2E_2 - 2E_3.$$

Therefore

$$c_1(Y).C + 2g - 2 = 4 - 2 - 2 + 0 - 2 = -2 < 0.$$

Similarly we can check for other curves $\widetilde{\Sigma_{i,j}}$. However, these curves are the unique effective curves in their cohomology classes, thus Theorem 2 does not apply. Theorem 1 does apply though, since $0 = c_1(Y).C \neq 2g - 2 = -2$. □

Proof. (Of Example 6) Let $p : Y \to \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ at e_1 and e_3. Let E_1 and E_3 be the exceptional divisors, and let $L_1 \subset E_1$ and $L_2 \subset E_2$ be generic lines. Let $H \subset \mathbb{P}^3$ be a generic hyperplane. Since $\Sigma_{0,1}$ contains e_3, the class of its strict transform $\widetilde{\Sigma_{0,1}}$ in Y is $H.H - L_3$. Because

$$c_1(Y) = p^*(c_1(\mathbb{P}^3)) - 2E_1 - 2E_3 = 4H - 2E_2 - 2E_3,$$

and the genus of $\widetilde{\Sigma_{0,1}}$ is $g = 0$, we have

$$c_1(Y).\widetilde{\Sigma_{0,1}} + 2g - 2 = (4H - 2E_2 - 2E_3)(H.H - L_3) - 2 = 4 - 2 - 2 = 0.$$

Therefore $\widetilde{\Sigma_{0,1}}$ does not satisfy condition 2) in Theorem 2, and it does not satisfy conditions 1) and 3) as well. But we can compute directly (as in part b) of the proof of Theorem 2) to show that the space Z, which is the blowup of Y at $\widetilde{\Sigma_{0,1}}$ satisfies Condition (A) in the proof of Theorem 2. Let F be the exceptional divisor of the blowup $q : Z \to Y$ and let $M \subset F$ be a fiber of $F \to \widetilde{\Sigma_{0,1}}$. Since $e_1 \in \Sigma_{0,3}$ and $\Sigma_{0,3} \cap \Sigma_{0,1} = e_2 \neq e_1, e_3$, the class of the strict transform $\widetilde{\Sigma_{0,3}}$ in Z is $H.H - L_1 - M$. Meanwhile

$$c_1(Z) = q^*(c_1(Y)) - F = 4H - 2E_1 - 2E_3 - F.$$

Therefore, since the genus of $\widetilde{\Sigma_{0,3}}$ is $g = 0$, it follows that $c_1(Z).\widetilde{\Sigma_{0,3}} + 2g - 2 = -1 < 0$. Moreover, $\widetilde{\Sigma_{0,3}}$ is not the only effective curve in its cohomology class (its cohomology class is the same as the class of the strict transform of a generic line passing to e_3 and intersecting $\Sigma_{0,1}$). Part d) of the proof of Theorem 2 implies that X satisfies Condition (A). Therefore any automorphism of X has zero entropy.

Alternatively, we can show that $\widetilde{\Sigma_{0,3}}$ satisfies condition 3) of Theorem 2. We let S be the strict transform of the hyperplane $\Sigma_0 = \{x_0 = 0\} \subset \mathbb{P}^3$. Then S contains $\widetilde{\Sigma_{0,3}}$ with multiplicity $\mu = 1$. The class of S is $S = H - E_1 - E_3 - F$. Therefore $\kappa = S.\widetilde{\Sigma_{0,3}} = -1$, and

$$2\kappa = -2 < -1 = c_1(Z).\widetilde{\Sigma_{0,3}} + 2g - 2 = \mu\gamma.$$

□

References

[1] T. Bayraktar, *On the automorphism group of rational manifolds,* arXiv: 1210.4651.

[2] T. Bayraktar and S. Cantat, *Constraints on automorphism groups of higher dimensional manifolds,* arXiv: 1212.3735.

[3] E. Bedford, *The dynamical degrees of a mapping,* arXiv: 1110.1741.

[4] E. Bedford and K.-H. Kim, *Pseudo-automorphisms of 3-space: periodicities and positive entropy in linear fractional recurrences,* arXiv: 1101.1614.

[5] E. Bedford and K.-H. Kim, *Dynamics of rational surface automorphisms: rotation domains,* American J. of Math, to appear.

[6] E. Bedford and K.-H. Kim, *Continuous families of rational surface automorphisms with positive entropy,* Mathematische Annalen 348 (2010), no. 3, 667–688.

[7] E. Bedford and K.-H. Kim, *Dynamics of rational surface automorphisms : linear fractional recurrences,* J. Geom. Anal. 19 (2009), no. 3, 553–583.

[8] J. Blanc, *Dynamical degrees of (pseudo)-automorphisms fixing cubic hypersurfaces,* arXiv: 1204.4256.

[9] S. Cantat, *Dynamique des automorphismes des surfaces K3,* Acta Math. 187:1 (2001), 1–57.

[10] S. Cantat and I. Dolgachev, *Rational surfaces with a large group of automorphisms,* Journal of the AMS, to appear.

[11] J.-P. Demailly, *Complex analytic and differential geometry,* Online book, version of Thursday 10 September 2009.

[12] J.-P. Demailly and M. Paun, *Numerical characterization of the Kähler cone of a compact Kähler manifold,* Ann. of Math. (2) 159 (2004), no. 3, 1247–1274.

[13] J. Déserti and J. Grivaux, *Automorphisms of rational surfaces with positive entropy,* arXiv: 1004.0656.

[14] J. Diller, *Cremona transformations, surface automorphisms, and plane cubics.* With an appendix by Igor Dolgachev. Michigan Math. J. 60 (2011), no. 2, 409–440.

[15] T-C. Dinh and N. Sibony, *Une borne supérieure pour l'entropie topologique d'une application rationnelle,* Ann. of Math. (2) 161 (2005), no. 3, 1637–1644.

[16] T-C. Dinh and N. Sibony, *Regularization of currents and entropy,* Ann. Sci. École Norm. Sup. (4) 37 (2004), no. 6, 959–971.

[17] T-C. Dinh and N. Sibony, *Groupes commutatifs d'automorphismes d'une variété kählérienne compacte,* Duke Math. J. 123 (2004), no. 2, 311–328.

[18] M. Gromov, *On the entropy of holomorphic maps,* Enseignement Math. 49 (2003), 217–235. Manuscript (1977).

[19] P. Griffiths and J. Harris, *Principles of algebraic geometry,* 1978, John Wiley and Sons, Inc.

[20] R. Hartshorne, *Algebraic geometry,* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[21] C. McMullen, *Dynamics with small entropy on projective K3 surfaces,* preprint.

[22] C. McMullen, *K3 surfaces, entropy and glue,* J. Reine Angew. Math. 658 (2011), 1–25.

[23] C. McMullen, *Dynamics on blowups of the projective plane,* Publ. Math. Inst. Hautes tudes Sci. No. 105 (2007), 49–89.

[24] C. McMullen, *Dynamics on K3 surfaces: Salem numbers and Siegel disks,* J. Reine Angew. Math. 545 (2002), 201–233.

[25] K. Oguiso and F. Perroni, *Automorphisms of rational manifolds of positive entropy with Siegel disks* , arXiv:0905.4257.

[26] K. Oguiso, *Free automorphisms of positive entropy on smooth Kähler surfaces,* arXiv:1202.2637.

[27] K. Oguiso, *The third smallest Salem number in automorphisms of K3 surfaces,* Algebraic geometry in East AsiaSeoul 2008, 331–360, Adv. Stud. Pure Math., 60, Math. Soc. Japan, Tokyo, 2010.

[28] K. Oguiso, *A remark on dynamical degrees of automorphisms of hyperkähler manifolds,* Manuscripta Math. 130 (2009), no. 1, 101–111.

[29] K. Oguiso, *Automorphisms of hyperkähler manifolds in the view of topological entropy,* Algebraic geometry, 173–185, Contemp. Math., 422, Amer. Math. Soc., Providence, RI, 2007.

[30] F. Perroni and D.-Q. Zhang, *Pseudo-automorphisms of positive entropy on the blowups of products of projective spaces,* arXiv:1111.3546.

[31] P. Reschke, *Salem numbers and automorphisms of complex surfaces,* Math Research Letters, to appear. arXiv:1202.5245.
[32] Y. Yomdin, *Volume growth and entropy,* Israel J. Math., 57 (1987), 285–300.
[33] D.-Q. Zhang, *Automorphism groups of positive entropy on minimal projective varieties,* Adv. Math. 225 (2010), no. 5, 2332–2340.
[34] D.-Q. Zhang, *The g-periodic subvarieties for an automorphism g of positive entropy on a compact Kähler manifold,* Adv. Math. 223 (2010), no. 2, 405–415.
[35] D.-Q. Zhang, *Dynamics of automorphisms on projective complex manifolds,* J. Differential Geom. 82 (2009), no. 3, 691–722.
[36] D.-Q. Zhang, *Automorphism groups and anti-pluricanonical curves,* Math. Res. Lett. 15 (2008), no. 1, 163–183.

Indiana University, Bloomington IN 47405
E-mail address: truongt@indiana.edu