UNIRATIONALITY OF UENO-CAMPANA'S THREEFOLD

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Abstract. We shall prove that the threefold studied in the paper " Remarks on an Example of K. Ueno" by F. Campana is unirational. This gives an affirmative answer to a question posed in the paper above and also in the book by K. Ueno, "Classification theory of algebraic varieties and compact complex spaces".

1. Introduction

Let k be any field of characteristic $\neq 2$ containing a primitive fourth root of unity $\sqrt{-1}$. We shall work over k unless otherwise stated. Let $[x : y : z]$ be the homogeneous coordinates of $\mathbb{P}^2$ and let

$$C := (y^2 z = x(x^2 - z^2)) \subset \mathbb{P}^2$$

be the harmonic elliptic curve, having an automorphism g of order 4 defined by

$$g^*(x : y : z) = (-x : \sqrt{-1}y : z)$$

whose quotient is $\mathbb{P}^1$. When k is the complex number field $\mathbb{C}$, we have

$$(C, g) \simeq (E_{\sqrt{-1}}, \sqrt{-1}) \ ,$$

where $E_{\sqrt{-1}} = \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$, the elliptic curve of period $\sqrt{-1}$ and $\sqrt{-1}$ is the automorphism induced by multiplication by $\sqrt{-1}$ on $\mathbf{C}$. This is because the complex elliptic curve with an automorphism of order 4 acting on the space of global holomorphic 1-forms as $\sqrt{-1}$ is unique up to isomorphism.

Let (C_j, g_i) $(j = 1, 2, 3)$ be three copies of (C, g). Let

$$Z = C_1 \times C_2 \times C_3 \ .$$

For simplicity, we denote the automorphism of Z defined by (g_1, g_2, g_3) by the same letter g. Then g is an automorphism of Z of order 4 and the quotient threefold

$$Y := (C_1 \times C_2 \times C_3)/\langle g \rangle$$

has 8 singular points of type $(1, 1, 1)/4$ and 28 singular points of type $(1, 1, 1)/2$. Let X be the blow up of Y at the maximal ideals of these singular points. Then X is a smooth projective threefold defined over k. In his paper [Ca12], F. Campana proved that X is a rationally connected threefold when $k = \mathbb{C}$. We shall call X the *Ueno-Campana's threefold*.

In [Ca12, Question 4], F. Campana asked if X is rational or unirational (at least over $\mathbb{C}$)? See also [Ue75, Page 208] for this Question and [OT13] for a relevant example and application to complex dynamics. The aim of this short note is to give an affirmative answer to this question:

Theorem 1.1. *Ueno-Campana's threefold X is unirational, i.e., there is a dominant rational map $\mathbb{P}^3 \cdots \to X$.*

We shall show that X is birationally equivalent to the Galois quotient of a conic bundle over $\mathbb{P}^2$ *with* a rational section, while X itself is birationally equivalent to a conic bundle over $\mathbb{P}^2$ *without* any rational section.

We are still working on the question whether X is a rational variety.

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2. Proof of Theorem (1.1)

The curves (C_i, g_i) $(i = 1, 2, 3)$ are birationally equivalent to (C_i^0, g_i), where C_i^0 is the curve in the affine space $\mathbb{A}^2 = \operatorname{Spec} k[X_i, Y_i]$, and g_i is the automorphism of C_i^0, defined by

$$Y_i^2 = X_i(X_i^2 - 1)\ ,\ g_i^* Y_i = \sqrt{-1} Y_i\ ,\ g_i^* X_i = -X_i\ .$$

The affine coordinate ring $k[C_i^0]$ of C_i^0 is

$$k[C_i^0] = k[X_i, Y_i]/(Y_i^2 - X_i(X_i^2 - 1)).,$$

WE set $x_i := X_i \operatorname{mod}(Y_i^2 - X_i(X_i^2 - 1)), y_i := Y_i \operatorname{mod}(Y_i^2 - X_i(X_i^2 - 1))$. We note that $y_i^2 = x_i(x_i^2 - 1)$, $g^* y_i = \sqrt{-1} y_i$, $g^* x_i = -x_i$ in $k[C_i^0]$.

Then $(Z = C_1 \times C_2 \times C_3, g = (g_1, g_2, g_3))$ is birationally equivalent to the affine threefold

$$V := C_1^0 \times C_2^0 \times C_3^0$$

with automorphism (g_1, g_2, g_3), which we denote by the same letter g, and with affine coordinate ring

$$k[V] = k[C_i^0] \otimes k[C_2^0] \otimes k[C_3^0] \text{ generated by } x_1, x_2, x_3, y_1, y_2, y_3\ .$$

The rational function field $k(Z)$ of Z is

$$k(Z) = k(V) = k(x_1, x_2, x_3, y_1, y_2, y_3)\ .$$

In both $k[V]$ and $k(Z)$, we have

$$\text{(I) } y_i^2 = x_i(x_i^2 - 1)\ ,$$

$$\text{(II) } g^* y_i = \sqrt{-1} y_i\ ,\ g^* x_i = -x_i\ .$$

Since X is birationally equivalent to $V/\langle g \rangle$, the rational function field $K(X)$ of X is identified with the invariant subfield $k(Z)^g$ of $k(Z)$, i.e.,

$$k(X) = k(Z)^g = \{f \in k(Z)\,|g^* f = f\}\ .$$

Consider the following elements in $k(Z)$:

$$\text{(III) } b_2 := \frac{x_2}{x_1}\ ,\ b_3 := \frac{x_3}{x_1}\ ,\ a_2 := \frac{y_2}{y_1}\ ,\ a_3 := \frac{y_3}{y_1}\ ,$$

$$\text{(IV) } u_1 := x_1^2\ ,\ w_1 := y_1^4\ ,\ \lambda_1 := x_1 y_1^2\ ,$$

and define the subfield L of $k(Z)$ by

$$L := k(b_2, b_3, a_2, a_3, u_1, w_1, \lambda_1)\ .$$

Here we used the fact that $x_1 \neq 0$, $y_1 \neq 0$ in $k(Z)$.

Lemma 2.1. *$k(X) = L$ in $k(Z)$.*

Proof. By (II) and (III), b_2, b_3, a_2, a_3, u_1, w_1, λ_1 are g-invariant. Hence

$$\text{(V) } L \subset k(X) \subset k(V)\ .$$

Note that $k(Z) = L(y_1)$. This is because

$$x_1 = \frac{\lambda_1}{y_1^2}\ ,\ x_2 = b_2x_1, x_3 = b_3x_1\ ,\ y_2 = a_2y_1\ ,\ y_3 = a_3y_1\ ,$$

by (III) and (IV). Since $y_1^4 = w_1$ and $w_1 \in k(Z)$, it follows that

$$\text{(VI) } [k(Z) : L] \le 4\ ,$$

where $[k(Z) : L]$ is the degree of the field extension $L \subset k(Z)$, i.e., the dimension of $k(Z)$ being naturally regarded as the vector space over L.

On the other hand, the group $\langle g\rangle \subset \mathrm{Gal}(k(Z)/k(X))$ is of order 4. Thus, by the fundamental theorem of Galois theory, we have that

$$\text{(VII) } [k(Z) : k(X)] = [K(Z) : k(Z)^g] = \mathrm{ord}\,(g) = 4\ .$$

The result now follows from (V), (VI), (VII). Indeed, by (V), we have

$$[k(Z) : L] = [k(Z) : k(X)][k(X) : L]\ .$$

On the other hand, $[k(Z) : L] \le 4$ by (VI), and $[k(X) : L] \ge 1$. Hence $[k(X) : L] = 1$ by (VII). This means that $L = k(X)$ in $k(Z)$, as claimed. □

Lemma 2.2. *$L = k(u_1, b_2, b_3, a_2, a_3)$ in $k(Z)$.*

Proof. Since $u_1, b_2, b_3, a_2, a_3 \in L$, it follows that $k(u_1, b_2, b_3, a_2, a_3) \subset L$. Let us show $L \subset k(u_1, b_2, b_3, a_2, a_3)$. For this, it suffices to show that $w_1, \lambda_1 \in k(u_1, b_2, b_3, a_2, a_3)$.

Recall by (I), $y_1^2 = x_1(x_1^2 - 1)$, Hence taking the square and using (VI), we obtain that

$$\text{(VIII) } w_1 = y_1^4 = x_1^2(x_1^2 - 1)^2 = u_1(u_1 - 1)^2\ .$$

Hence $w_1 \in k(u_1, b_2, b_3, a_2, a_3)$. From $y_1^2 = x_1(x_1^2 - 1)$ again, we have that

$$\text{(IX) } \lambda_1 = x_1y_1^2 = x_1^2(x_1^2 - 1) = u_1(u_1 - 1)\ .$$

Hence $\lambda_1 \in k(u_1, b_2, b_3, a_2, a_3)$ as well. □

Lemma 2.3. *Let $j = 2, 3$. Then, $a_j^2 - b_j \ne 0$ in both $k(Z)$ and $k(X)$.*

Proof. By using (I), we obtain that

$$\text{(X) } a_j^2 - b_j = \frac{y_j^2}{y_1^2} - \frac{x_j}{x_1} = \frac{x_j(x_j^2 - 1)}{x_1(x_1^2 - 1)} - \frac{x_j}{x_1} = \frac{x_j}{x_1}\left(\frac{x_j^2 - 1}{x_1^2 - 1} - 1\right)\ ,$$

in $k(Z)$. Recall that $x_i \ne 0$ in $k(Z)$. Thus, if $a_j^2 - b_j = 0$ in $k(Z)$, then we would have $(x_j^2 - 1)/(x_1^2 - 1) = 1$ in $K(Z) = k(V)$ from the equality above, and therefore, $x_j = \pm x_1$ in $k[V]$. However, this contradicts to the fact that x_1 is identically 0 on the set of $\overline{k}$-valued points $(\{0\} \times C_2 \times C_3)(\overline{k})$ but $\pm x_j$ $(j = 2, 3)$ are not identically 0 on it. This contradiction implies that $a_j^2 - b_j \ne 0$ in $k(Z)$. Since $a_j^2 - b_j \in k(Z)^g = k(X)$ and $k(X)$ is a subfield of $k(Z)$, it follows that $a_j^2 - b_j \ne 0$ in $k(X)$ as well. □

Proposition 2.4. *$k(X) = L = k(b_2, b_3, a_2, a_3)$ in $k(Z)$. More precisely, in $k(Z)$, we have*

$$\text{(XI)}\ u_1 = \frac{a_2^2 - b_2}{a_2^2 - b_2^3} = \frac{a_3^2 - b_3}{a_3^2 - b_3^3}\ .$$

Proof. By Lemma (2.1, 2.2), it suffices to show the equality (X) in $k(Z)$. Observe that, for $j = 2, 3$:

$$y_j^2 = x_j(x_j^2 - 1) \Leftrightarrow y_1^2 a_j^2 = x_1 b_j (x_1^2 b_j^2 - 1)$$

hence multiplication by x_1 yields

$$x_1^2 b_j (x_1^2 b_j^2 - 1) = x_1 y_1^2 a_j^2 = x_1^2 (x_1^2 - 1) a_j^2,$$

and dividing by x_1^2 and observing that $u_1 = x_1^2$ we obtain

$$b_j(u_1 b_j^2 - 1) = (u_1 - 1) a_j^2$$

i.e.,

$$(**)\ u_1(a_j^2 - b_j^3) = a_j^2 - b_j.$$

Using the previous lemma we obtain $(a_j^2 - b_j^3) \neq 0$, so we can divide and obtain (XI). □

Proposition 2.5. *X is birationally equivalent to the affine hypersurface H in $\mathbb{A}^4 = \operatorname{Spec} k[a, b, \alpha, \beta]$, defined by*

$$(a^2 - b)(\alpha^2 - \beta^3) = (\alpha^2 - \beta)(a^2 - b^3)\ ,$$

or equivalently defined by

$$a^2 \beta (1 - \beta^2) = \alpha^2 b (1 - b^2) + b\beta(b^2 - \beta^2)\ ,$$

Proof. By Lemma (2.1) and Proposition (2.4), $k(X) = k(a_2, a_3, b_2, b_3)$ in $k(Z)$, with a relation

$$\text{(XII)}\ (a_2^2 - b_2)(a_3^2 - b_3^3) = (a_3^2 - b_3)(a_2^2 - b_2^3)\ .$$

Expanding both sides and subtracting then the common term $a_2^2 a_3^2$, we obtain

$$-a_2^2 b_3^3 - b_2 a_3^2 + b_2 b_3^3 = -a_3^2 b_2^3 - b_3 a_2^2 + b_3 b_2^3\ .$$

Solving this relation in terms of a_2, we obtain that

$$\text{(XIII)}\ a_2^2 b_3 (1 - b_3^2) = a_3^2 b_2 (1 - b_2^2) + b_2 b_3 (b_2^2 - b_3^2)\ .$$

Since $b_3 = x_3/x_1$ is not a constant in $k(Z)$, it follows that $b_3(1 - b_3^2) \neq 0$ in $k(Z)$, whence also not 0 in $k(X)$. Thus

$$\text{(XIV)}\ a_2^2 = \frac{a_3^2 b_2 (1 - b_2^2) + b_2 b_3 (b_2^2 - b_3^2)}{b_3 (1 - b_3^2)}\ .$$

Therefore a_2 is algebraic over $k(a_3, b_2, b_3)$ of degree at most 2. Since X is of dimension 3 over k, it follows that a_3, b_2, b_3 form a transcendence basis of $k(X)$ over k. Thus, the subring $k[a_3, b_2, b_3]$ of $k(X)$ is isomorphic to the polynomial ring over k of Krull-dimension 3. Moreover, the right hand side of (XIV) is not a square in $k(a_3, b_2, b_3)$. Indeed, the multiplicity of b_3 in the denominator is 1 while the numerator is not in k and the multiplicity of b_3 in the numerator is 0. Thus the equation (XIV) is the minimal polynomial of a_2 over $k(a_3, b_2, b_3)$. Hence X is birationally equivalent to the double cover of $\mathbb{A}^3 = \operatorname{Spec} k[a_3, b_2, b_3]$, defined by (XIV). This means that X is birationally equivalent to the hypersurface in the affine space $\mathbb{A}^4 = \operatorname{Spec} k[a, \alpha, b, \beta]$, defined by (XIV) or equivalently defined by (XIII) or by (XII), in which (a_2, a_3, b_2, b_3) are replaced by (a, α, b, β). □

Corollary 2.6. *Let $H \subset \mathbb{A}^4 = \operatorname{Spec} k[a, \alpha, b, \beta]$ be the same as in Proposition (2.5). Consider the affine plane $\mathbb{A}^2 = \operatorname{Spec} k[b, \beta]$ and the natural projection*

$$\pi : \mathbb{A}^4 \to \mathbb{A}^2$$

defined by

$$(a, b, \alpha, \beta) \mapsto (b, \beta) \ .$$

Then the natural restriction map

$$p := \pi | H : H \to \mathbb{A}^2$$

is a conic bundle over $\mathbb{A}^2$. In particular, the graph Γ of the rational map $\tilde{p} : X \cdots \to \mathbb{P}^2$ naturally induced by p forms a conic bundle on Γ over $\mathbb{P}^2$. We note that Γ is projective and birationally equivalent to X over k.

Proof. The fibre $\pi^{-1}(\eta)$ of π over the generic point $\eta \in \mathbb{A}^2 = \operatorname{Spec} k[b, \beta]$ is the affine space $\mathbb{A}^2_\eta = \operatorname{Spec} k(b, \beta)[a, \alpha]$ defined over $\kappa(\eta) = k(b, \beta)$. Thus by the second equation in Proposition (2.5), the generic fibre $X_\eta := (\pi|H)^{-1}(\eta)$ is the conic in $\mathbb{A}^2_\eta$, defined by

$$a^2\beta(1 - \beta^2) = \alpha^2 b(1 - b^2) + b\beta(b^2 - \beta^2) \ .$$

This implies the result. □

Remark 2.7. The conic X_η in the proof of Proposition (2.6) has no rational point over $\kappa(\eta) = k(b, \beta)$, i.e., the set $X_\eta(k(b, \beta))$ is empty.

Proof. Suppose to the contrary that $(a(b, \beta), \alpha(b, \beta)) \in X_\eta(k(b, \beta))$. We can write

$$a(b, \beta) = \frac{P(b, \beta)}{Q(b, \beta)} \ , \ \alpha(b, \beta) = \frac{R(b, \beta)}{Q(b, \beta)} \ ,$$

where $P(b, \beta), Q(b, \beta), R(b, \beta) \in k[b, \beta]$ with no non-constant common factor, possibly after replacing the denominators by their product. Then substituting the above into the equation of X_η and clearing the denominator, we would have the following identity in $k[b, \beta]$:

$$P(b, \beta)^2\beta(1 - \beta^2) = R(b, \beta)^2 b(1 - b^2) + Q(b, \beta)^2 b\beta(b^2 - \beta^2) \ .$$

Since $k[b, \beta]$ is a polynomial ring, in particular, it is a UFD, it would follow that $P(b, \beta)$ is divisible by b and $R(b, \beta)$ is divisible by β in $k[b, \beta]$. Thus $P(b, \beta) = P_1(b, \beta)b$ and $R(b, \beta) = R_1(b, \beta)\beta$ for some $P_1(b, \beta), R_1(b, \beta) \in k[b, \beta]$. Substituting these two into the equality above and dividing by $b\beta \neq 0$, it follows that

$$P_1(b, \beta)^2 b(1 - \beta^2) = R_1(b, \beta)^2\beta(1 - b^2) + Q(b, \beta)^2(b^2 - \beta^2) \ .$$

Substitute $b = 0$ into this equation: we obtain $R_1(0, \beta)^2\beta + Q(0, \beta)^2\beta^2 = 0$, which implies that $R_1(0, \beta) = Q(0, \beta) = 0$. This means that both $R_1(b, \beta)$ and $Q(b, \beta)$ are divisible by

b. Similarly, if we substitute $\beta = 0$ into the above equation we find that both $P_1(b, \beta)$ and $Q(b, \beta)$ are divisible by β. Thus we can write

$$P_1(b,\beta) = \beta P_2(b,\beta),\ R_1(b,\beta) = bR_2(b,\beta),\ Q(b,\beta) = b\beta Q_2(b,\beta),$$

where $P_2(b,\beta),\ R_2(b,\beta),\ Q(b,\beta) \in k[b,\beta]$. But this implies that all $P(b,\beta), Q(b,\beta), R(b,\beta)$ are divisible by $b\beta$, a contradiction. □

The next corollary completes the proof of Theorem (1.1):

Corollary 2.8. *Let* $H \subset \mathbb{A}^4 = \operatorname{Spec} k[a,\alpha,b,\beta]$*,* $p : H \to \mathbb{A}^2 = \operatorname{Spec} k[b,\beta]$ *be the same as in Proposition 2.5 and Corollary 2.6. Consider another affine space* $\operatorname{Spec} k[s,t]$ *and the (finite Galois) morphism of degree* 4

$$f : \operatorname{Spec} k[s,t] \to \operatorname{Spec} k[b,\beta]$$

defined by

$$f^*b = s^2\ ,\ f^*\beta = t^2\ .$$

Consider then the fibre product

$$Q := H \times_{\operatorname{Spec} k[b,\beta]} \operatorname{Spec} k[s,t]$$

and the natural second projection $p_2 : Q \to \operatorname{Spec} k[s,t]$*. Then* p_2 *is a conic bundle with a rational section and* Q *is a rational threefold. In particular,* H*, hence* X*, is unirational.*

Proof. Recall that H is the hypersurface in $\operatorname{Spec} k[a,b,\alpha,\beta]$ defined by

$$a^2\beta(1-\beta^2) = \alpha^2 b(1-b^2) + b\beta(b^2-\beta^2)\ ,$$

or equivalently by

$$(a^2-b)(\alpha^2-\beta^3) = (\alpha^2-\beta)(a^2-b^3)\ .$$

Thus, by definition of the fibre product, Q is a hypersurface in the affine space $\mathbb{A}^4 = \operatorname{Spec} k[a,\alpha,s,t]$, defined by

$$a^2t^2(1-t^4) = \alpha^2 s^2(1-s^4) + s^2t^2(s^4-t^4)\ ,$$

or equivalently by

$$(a^2-s^2)(\alpha^2-t^6) = (\alpha^2-t^2)(a^2-s^6)\ .$$

Then the natural projection $p_2 : Q \to \operatorname{Spec} k[s,t]$ is a conic bundle with generic fibre

$$Q_{\eta'} = (a^2t^2(1-t^4) = \alpha^2 s^2(1-s^2) + s^2t^2(s^4-t^4)) \subset \operatorname{Spec} k(s,t)[a,\alpha] = \mathbb{A}^2_{\eta'}\ ,$$

where η' is the generic point of $\operatorname{Spec} k[s,t]$. Then $Q_{\eta'}$ has a rational point $(a,\alpha) = (s,t) \in Q(k(s,t))$ over $\kappa(\eta') = k(s,t)$. Hence $Q_{\eta'}$ is isomorphic to $\mathbb{P}^1_{\eta'}$ over $k(s,t)$. Thus, denoting the affine coordinate of $\mathbb{P}^1_{\eta'}$ by v, we obtain that

$$k(Q) = k(s,t)(Q_{\eta'}) \simeq k(s,t)(\mathbb{P}^1_{\eta'}) = k(s,t)(v) = k(s,t,v)\ .$$

Since Q is of dimension 3 over k, it follows that s, t, v are algebraically independent over k. Hence, $k(Q)$ is isomorphic to the rational function field of $\mathbb{P}^3$ over k. Hence Q is a rational threefold over k, i.e., birationally equivalent to $\mathbb{P}^3$ over k. Since the natural morphism $p_1 : Q \to H$, i.e., the first projection morphism in the fibre product, is a finite dominant morphism of degree 4, Q is birational to $\mathbb{P}^3$ and H is birationally equivalent to X, all over

k, we obtain a rational dominant map $q : \mathbb{P}^3 \cdots \to X$ over k, from the natural projection $p_1 : Q \to H$. Hence X is unirational. □

References

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