THE SIMPLICITY OF THE FIRST SPECTRAL RADIUS OF A MEROMORPHIC MAP

TUYEN TRUNG TRUONG

Abstract. Let X be a compact Kähler manifold and let $f : X \to X$ be a dominant rational map which is 1-stable. Let λ_1 and λ_2 be the first and second dynamical degrees of f. If $\lambda_1^2 > \lambda_2$, then we show that λ_1 is a simple eigenvalue of $f^* : H^{1,1}(X) \to H^{1,1}(X)$, and moreover the unique eigenvalue of modulus $> \sqrt{\lambda_2}$. A variant of the result, where we consider the first spectral radius in the case the map f may not be 1-stable, is also given. An application is stated for bimeromorphic selfmaps of 3-folds. For another application, we estimate the second dynamical degree of a class of birational maps which arises in lattice statistical mechanics and is related to matrix inverses.

1. Introduction

Let X be a compact Kähler manifold of dimension k with a Kähler form ω_X, and let $f : X \to X$ be a dominant meromorphic map. For $0 \leq p \leq k$, the p-th dynamical degree $\lambda_p(f)$ of f is defined as follows

$$\lambda_p(f) = \lim_{n\to\infty} (\int_X (f^n)^*(\omega_X^p) \wedge \omega_X^{k-p})^{1/n} = \lim_{n\to\infty} r_p(f^n)^{1/n},$$

where $r_p(f^n)$ is the spectral radius of the linear map $(f^n)^* : H^{p,p}(X) \to H^{p,p}(X)$ (See Russakovskii-Shiffman [40] for the case where $X = \mathbb{P}^k$, and Dinh-Sibony [19][18] for the general case. In the case of complex projective manifolds there was also an approach by Friedland [23]; however, that approach contains a gap as observed in Guedj [29].). The dynamical degrees are log-concave, in particular $\lambda_1(f)^2 \geq \lambda_2(f)$. In the case $f^* : H^{2,2}(X) \to H^{2,2}(X)$ preserves the cone of psef classes (i.e. those $(2,2)$ cohomology classes which can be represented by positive closed $(2,2)$ currents), then we have an analog $r_1(f)^2 \geq r_2(f)$ (see Theorem 1.2).

The present paper concerns the first dynamical degree $\lambda_1(f)$ and more generally the first spectral radius $r_1(f)$. We will say that f is 1-stable if for any $n \in \mathbb{N}$, $(f^n)^* = (f^*)^n$ on $H^{1,1}(X)$ (the first use of this notion appeared in the paper Fornaess-Sibony [22] in the case of rational selfmaps of projective spaces). When f is 1-stable, we have $\lambda_1(f) = r_1(f)$. There are many examples of 1-stable maps, for example those which are pseudo-automorphisms.

The first main result of this paper is the following

Theorem 1.1. *Let X be a compact Kähler manifold of dimension k, and let $f : X \to X$ be a dominant meromorphic map which is* 1*-stable. Assume that $\lambda_1(f)^2 > \lambda_2(f)$. Then $\lambda_1(f)$*

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is a simple eigenvalue of $f^* : H^{1,1}(X) \to H^{1,1}(X)$*. Further,* $\lambda_1(f)$ *is the only eigenvalue of modulus greater than* $\sqrt{\lambda_2(f)}$*.*

Theorem 1.1 answers Question 3.3 in Guedj [28]. It was known when f is holomorphic (this case is very easy, due to the fact that $f^*(\theta_1.\theta_2) = f^*(\theta_1).f^*(\theta_2)$, hence if $f^*(\theta_{1,2}) = \lambda_1(f)\theta_{1,2}$ and $\lambda_1(f)^2 > \lambda_2(f)$ then $\theta_1.\theta_2 = 0$), see e.g. Cantat-Zeghib [12] where the case of holomorphic maps of 3-folds is explicitly stated. An immediate consequence of Theorem 1.1 is that if f is 1-stable and $\lambda_1(f)^2 > \lambda_2(f)$, then the "degree growth" of f satisfies $deg(f^n) = c\lambda_1(f)^n + O(\tau^n)$ for some constants $c > 0$ and $\tau < \lambda_1(f)$. In the case X is a surface, the same estimate for the degree growth was obtained in Boucksom-Favre-Jonsson [13] where the condition f is 1-stable is not needed. The conclusion of Theorem 1.1 that $\lambda_1(f)$ is simple is very helpful in constructing Green currents and proving equi-distribution properties toward it (see e.g. Guedj [28], Diller-Guedj [16] and Bayraktar [3]).

When X is a compact Kähler surface, Diller-Favre [15] proved a stronger conclusion than that of Theorem 1.1 where the condition of 1-stability is dropped. The following variant of Theorem 1.1 gives a generalization of Diller and Favre's result to higher dimensions. Recall that $r_1(f)$ is the spectral radius of $f^* : H^{1,1}(X) \to H^{1,1}(X)$ and $r_2(f)$ is the spectral radius of $f^* : H^{2,2}(X) \to H^{2,2}(X)$.

Theorem 1.2. *Let* X *be a compact Kähler manifold, and let* $f : X \to X$ *be a dominant meromorphic map. Assume that* $f^* : H^{2,2}(X) \to H^{2,2}(X)$ *preserves the cone of psef classes. Then*

1) We have $r_1(f)^2 \geq r_2(f)$*.*

2) Assume moreover that $r_1(f)^2 > r_2(f)$*. Then* $r_1(f)$ *is a simple eigenvalue of* $f^* : H^{1,1}(X) \to H^{1,1}(X)$*. Further,* $r_1(f)$ *is the only eigenvalue of modulus greater than* $\sqrt{r_2(f)}$*.*

As a consequence, we obtain the following

Corollary 1.3. *Let* X *be a compact Kähler manifold of dimension* 3*. Let* $f : X \to X$ *be a bimeromorphic map such that both* f *and* f^{-1} *are* 1*-stable. Assume moreover that* $\lambda_1(f) > 1$*. Then either* f *or* f^{-1} *satisfies the conclusions of Theorem 1.1.*

Proof. Observe that $\lambda_1(f^{-1}) = \lambda_2(f)$ and $\lambda_2(f^{-1}) = \lambda_1(f)$. Hence when $\lambda_1(f) > 1$, at least one of the following conditions hold: $\lambda_1(f)^2 > \lambda_2(f)$ and $\lambda_1(f^{-1})^2 > \lambda_2(f^{-1})$. □

Corollary 1.3 can be applied to pseudo-automorphisms $f : X \to X$ of a 3-fold X with $\lambda_1(f) > 1$. By definition (see e.g. [21]), a bimeromorphic map $f : X \to X$ is pseudo-automorphic if there are subvarieties V, W of codimension at least 2 so that $f : X - V \to X - W$ is biholomorphic. If X has dimension 3, then any pseudo-automorphism $f : X \to X$ is both 1-stable and 2-stable (see Bedford-Kim [4]). The first examples of pseudo-automorphisms with first dynamical degree larger than 1 on blowups of $\mathbb{P}^3$ were given in [4], by studying linear fractional maps in dimension 3. There are now several other examples in any dimension (see e.g. Perroni-Zhang [37], Blanc [9] and Oguiso [36]).

While the first dynamical degrees can be computed explicitly in various examples (e.g. by making a map 1-algebraic stable and computing the spectral radius of the action on $H^{1,1}$), it is much more difficult to compute the second dynamical degrees (for instance, not like the case of 1-algebraic stability, currently there is no general criterion to check whether a map is 2-algebraic stable). In this aspect, Theorem 1.1 can be used to estimate the second dynamical degrees. We illustrate this here for a class of birational maps which arises in lattice statistical mechanics and is related to matrix inverses (see Boukraa-Maillard [11], Bellon-Viallet [8], Boukraa-Hassani-Maillard [10], Auriac-maillard-viallet [1][2], Bedford-Kim [5][6], Preissmann-Auriac-Maillard [38], and [7] and [42]).

Corollary 1.4. *Let $q \geq 5$ be an integer. Let $\mathcal{M}_q$ be the space of $q \times q$ matrices with complex coefficients. Let $I : \mathcal{M}_q \to \mathcal{M}_q$ be the inverse map $I(x) = (x)^{-1}$ for $x \in \mathcal{M}_q$, and let $J : \mathcal{M}_q \to \mathcal{M}_q$ be the Hadamard inverse $J(x) = (1/x_{i,j})$ for $x = (x_{i,j}) \in \mathcal{M}_q$. The map $K = I \circ J$ defines a birational map $K : \mathbb{P}(\mathcal{M}_q) \to \mathbb{P}(\mathcal{M}_q)$, here we identify $\mathbb{P}(\mathcal{M}_q)$ with the projective space $\mathbb{P}^{q^2-1}$. Let δ_1 be the largest root of the equation $t^2 - (q^2 - 4q + 2)t + 1 = 0$, and let δ_2 be the largest root of the equation $t^2 - (q - 2)t + 1 = 0$. Then*

$$\lambda_1(K) = \delta_1,$$
$$\delta_2^2 \leq \lambda_2(K) \leq \delta_1^2.$$

In particular, the growth of $\lambda_2(K)$, as a function of q, is of order at least q^2 and at most q^4.

Proof. By results in [7], there is a finite composition of blowups along smooth centers $\pi : X \to \mathbb{P}(\mathcal{M}_q)$ such that the lifting map $K_X = \pi^{-1} \circ K \circ \pi : X \to X$ is 1-algebraic stable. Moreover the characteristic of the pullback $K_X^* : H^{1,1}(X) \to H^{1,1}(X)$ is

$$P(t)Q(t)(t-1)^{q^2-q+2}(t+1)^{q^2-3q+2},$$

where $P(t) = t^2 - (q^2 - 4q + 2)t + 1$ and $Q(t) = (t^2 - (q-2)t + 1)(t^2 + (q-2)t + 1)$. Hence the spectral radius of $K_X^* : H^{1,1}(X) \to H^{1,1}(X)$ is δ_1, and hence $\lambda_1(K) = \lambda_1(K_X) = \delta_1$. We finish the proof by estimating $\lambda_2(K)$. First, by the log-concavity of $\lambda_1(K)$ we have $\delta_1^2 \geq \lambda_2(K)$. If $\lambda_1(K_X)^2 = \delta_1^2 > \lambda_2(K) = \lambda_2(K_X)$ then Theorem 1.1 applied to the map K_X implies that any other eigenvalue of $K_X^* : H^{1,1}(X) \to H^{1,1}(X)$ is $\leq \sqrt{\lambda_2(K_X)} = \sqrt{\lambda_2(K)}$. In particular, we have $\lambda_2(K) \geq \delta_2^2$. □

The key tools in the proofs of Theorems 1.1 and 1.2 are the Hodge index theorem (Hodge-Riemann bilinear relations), Hironaka's elimination of indeterminacies for meromorphic maps, and a pull-push formula for blowups along smooth centers. Section 2 is devoted to the proofs of Theorems 1.1 and 1.2.

All of the above results have analogs in the algebraic setting, where X is a projective manifold over an algebraic closed field of characteristic zero, and $f : X \to X$ is a rational map. This will be done in a forthcoming paper. Further applications of the results in the current paper are being explored in ongoing projects. These include pseudo-automorphisms of dimension at most 4, automorphisms of complex 3-tori, and dynamics over non-Archimedean fields.

2. Proofs of Theorems 1.1 and 1.2

Let X and Y be compact Kähler manifolds and let $h : X \to Y$ be a dominant meromorphic map. By Hironaka's elimination of indeterminacies (see e.g. Corollary 1.76 in Kollár [34] and Theorem 7.21 in Harris [31] for the case X is projective, and see Hironaka [32] and Moishezon [35] for the general case), there is a compact Kähler manifold Z, a map $\pi : Z \to X$ which is a finite sequence of blowups along smooth centers, and a surjective holomorphic map $g: Z \to Y$, so that $h = g \circ \pi^{-1}$. (Since the analytic case of Hironaka's elimination of indeterminacies is less known, we give here a sketch of how to prove it, cf. the paper Ishii-Milman [33] for related ideas. We thank Pierre Milman for his generous help with this. Consider Γ a resolution of singularities of the graph Γ_h, and let $p, \gamma : \Gamma \to X, Y$ be the induced holomorphic maps. In particular $p: \Gamma \to X$ is a modification. By global Hironaka's flattening theorem, we can find a finite sequence of blowups $\pi : X' \to X$ along smooth centers, and let $\pi_\Gamma : \Gamma' \to \Gamma$ be the corresponding blowup along the ideals which are pullbacks by p of the ideals of the centers of the blowup π, so that the induced map $p' : \Gamma' \to X'$ is still holomorphic, bimeromorphic and flat. A priori, Γ' may be singular. But a holomorphic, bimeromorphic and flat map must actually be a biholomorphic map. Therefore, Γ' is also smooth, p' is biholomorphic, and the holomorphic maps $\pi : Z = X' \to X$ and $g = \gamma \circ \pi_\Gamma \circ p'^{-1} : Z = X' \to Y$ are what needed.)

For our purpose here, it is important to study the blowups whose center is a smooth submanifold of codimension exactly 2. We consider first the case of a single blowup. We use the conventions that if W is a subvariety then $[W]$ denotes the current of integration along W, and if T is a closed current then $\{T\}$ denotes its cohomology class (for the case $T = [W]$ where W is a subvariety, we write $\{W\}$ instead of $\{[W]\}$ for convenience). For two cohomology classes u and v, we denote by $u.v$ the cup product.

We have the following pull-push formulas for a single blowup (a more precise version of this for birational surface maps was given in [15])

Lemma 2.1. *Let X be a compact Kähler manifold of dimension k. Let $\pi : Z \to X$ be a blowup of X along a smooth submanifold $W = \pi(E)$ of codimension exactly 2. Let E be the exceptional divisor and let L be a general fiber of π.*

i) There is a constant $c_E > 0$ so that

$$(\pi)_*(\{E\}.\{E\}) = -c_E\{W\}.$$

ii) If α is a closed smooth $(1,1)$ form with complex coefficients on Z then

$$\pi^*(\pi)_*(\alpha) = \alpha + (\{\alpha\}.\{L\})[E].$$

iii) If α is a closed smooth $(1,1)$ form with complex coefficients on Z then

$$(\pi)_*(\alpha \wedge [E]) = c_E(\{\alpha\}.\{L\})[W].$$

iv) If α is a closed smooth $(1,1)$ form with complex coefficients on Z then

$$(\pi)_*((\pi)^*(\pi)_*(\alpha) \wedge \overline{\alpha}) - (\pi)_*(\alpha \wedge \overline{\alpha}) = c_E|\{\alpha\}.\{L\}|^2[W].$$

Remarks:

1) If X is projective, then $c_E = 1$ in the lemma (see Lemma **??**). We thank Charles Favre for showing this to us.

2) Lemma 2.1 i), iii), iv) and v) are trivially true when the center of blowup $W = \pi_1(E)$ has codimension at least 3. For example, then in i) we have $\pi_*(\{E\}.\{E\}) = 0$. In fact, by the same argument as in the proof of i) below, the cohomology class $\pi_*(\{E\}.\{E\})$ can be represented by a difference of two positive closed $(2,2)$ currents supported in $W = \pi(E)$. Since W has codimension at least 3, it follows that $\pi_*(\{E\}.\{E\}) = 0$.

Proof. i) By Demailly's regularization for positive closed $(1,1)$ currents (see Demailly [14], and also Dinh-Sibony [18]), there are positive closed smooth $(1,1)$ forms α_n,β_n of bounded masses so that $\alpha_n-\beta_n$ weakly converges to the current of integration $[E]$. Let α and β be any cluster points of the currents $\alpha_n\wedge[E]$ and $\beta_n\wedge[E]$, then α and β are positive closed $(2,2)$ currents with support in E and in cohomology $\{\alpha-\beta\}=\{E\}.\{E\}$. Therefore $\pi_*(\{E\}.\{E\})$ can be represented by the difference $\pi_*(\alpha)-\pi_*(\beta)$ of two positive closed $(2,2)$ currents $\pi_*(\alpha)$ and $\pi_*(\beta)$. Each of the latter has support in $W=\pi(E)$, hence since W has codimension exactly 2, each of them must be a multiple of the current of integration $[W]$ by the support theorem for normal currents. We infer

$$\pi_*(\{E\}.\{E\})=-c_E\{W\},$$

for a constant c_E. It remains to show that $c_E>0$. To this end, we let ω_X be a Kähler form on X. Then we get

$$\{E\}.\{E\}.\{\pi^*(\omega_X^{k-2})\}=(\pi)_*(\{E\}.\{E\}).\{\omega_X^{k-2}\}=-c_E\{W\}.\{\omega_X^{k-2}\}.$$

Since $\{W\}.\{\omega_X^{k-2}\}=\{[W]\wedge\omega_X^{k-2}\}$ is a positive number (equal the mass of W), to show that $c_E>0$ it suffices to show that $\{E\}.\{E\}.\{\pi^*(\omega_X^{k-2})\}<0$. If we can show that $\{E\}.\{\pi^*(\omega_X^{k-2})\}=a\{L\}$ for some constant $a>0$ then $\{E\}.\{E\}.\{\pi^*(\omega_X^{k-2})\}=a\{E\}.\{L\}=-a<0$ as wanted. To this end, first we observe that $\{E\}.\{\pi^*(\omega_X^{k-2})\}=a\{L\}$ for some constant a, because $H^{k-1,k-1}(Z)$ is generated by $\pi_1^*H^{k-1,k-1}(X)$ and $\{L\}$, and by the projection formula $(\pi)_*(\{E\}.\{\pi^*(\omega_X^{k-2})\})=(\pi)_*(\{E\}).\{\omega_X^{k-2})\}=0$. The constant a then must be positive, because if $\iota_E:E\to X$ and $\iota:W\to Y$ are the inclusion maps and $\pi_E:E\to W$ is the projection then

$$\begin{aligned}\{E\}.\{\pi^*(\omega_X^{k-2})\} &= (\iota_E)_*(\iota_E^*\pi^*\{\omega_X^{k-2}\})\\ &= (\iota_E)_*(\pi_E^*(\omega_X^{k-2}|_W)).\end{aligned}$$

The cohomology class $\{\omega_X^{k-2}\}|_W$ is a positive multiple of the class of a point, and since the map π_E is a fibration it follows that $\pi_E^*(\{\omega_X^{k-2}\}|_W)$ is a positive multiple of a fiber, and therefore $\{E\}.\{\pi^*(\omega_X^{k-2})\}$ is a positive multiple of a fiber.

ii) This is a standard result using $\{E\}.\{L\}=-1$ (see also iii) below).

iii) Since $(\pi)_*(\alpha\wedge[E])$ is a normal $(2,2)$ current with support in $W=\pi(E)$ which is a subvariety of codimension 2 in X, by support theorem it follows that there is a constant c such that $(\pi)_*(\alpha\wedge[E])=c[W]$. It is clear that c depends only on the cohomology class of $(\pi)_*(\alpha\wedge[E])$. Since $H^{1,1}(Z)$ is generated by $\pi^*(H^{1,1}(X))$ and $\{E\}$, we can write $\{\alpha\}=a\pi^*(\beta)+b\{E\}$ where $\beta\in H^{1,1}(X)$. Then using i) and the projection formula we obtain

$$\begin{aligned}(\pi)_*\{\alpha\wedge[E]\} &= (\pi)_*(\{\alpha\}.\{E\})=b(\pi)_*(\{E\}.\{E\})\\ &= -bc_E\{\pi(E)\}.\end{aligned}$$

Therefore $c=-bc_E$. The constant $-b$ can be computed as follows

$$\{\alpha\}.\{L\}=(a\pi^*(\beta)+b\{E\}).\{L\}=b\{E\}.\{L\}=-b.$$

Hence $c=(\{\alpha\}.\{L\})c_E$ as claimed.

iv) We have

$$\begin{aligned}(\pi)_*(\pi^*(\pi)_*(\alpha)\wedge\overline{\alpha}) &= (\pi)_*((\alpha+(\{\alpha\}.\{L\})[E])\wedge\overline{\alpha})\\ &= (\pi)_*(\alpha\wedge\overline{\alpha})+(\{\alpha\}.\{L\})(\pi)_*([E]\wedge\overline{\alpha})\\ &= (\pi)_*(\alpha\wedge\overline{\alpha})+c_E|\{\alpha\}.\{L\}|^2[\pi(E)].\end{aligned}$$

Thus iv) is proved. □

In particular, Lemma 2.1 shows that for a single blowup $\pi : Z \to X$, if α is a closed smooth $(1,1)$ form with complex coefficients then $(\pi)_*((\pi)^*(\pi)_*(\alpha)\wedge\overline{\alpha})-(\pi)_*(\alpha\wedge\overline{\alpha})$ is a positive closed $(2,2)$ current. (If the center of blowup W has codimension exactly 2 then this follows from Lemma 2.1 iv), while if W has codimension at least 3 then $(\pi)_*((\pi)^*(\pi)_*(\alpha)\wedge\overline{\alpha})-(\pi)_*(\alpha\wedge\overline{\alpha})=0$ as observed in the remarks after the statement of Lemma 2.1.) It follows that if $u\in H^{1,1}(Z)$ is a cohomology class with complex coefficients, then $\pi_*(u).\pi_*(\overline{u})-\pi_*(u.\overline{u})$ is a psef class, that is can be represented by a positive closed $(2,2)$ current. In fact, let α be a closed smooth $(1,1)$ form representing u. Then, $(\pi)_*(u.\overline{u})$ is represented by $(\pi)_*(\alpha\wedge\overline{\alpha})$, and by the projection formula $(\pi)_*(u).(\pi)_*(\overline{u})$ is represented by $(\pi)_*(\pi^*(\pi)_*(\alpha)\wedge\overline{\alpha})$. Hence from iv), we infer that $\pi_*(u).\pi_*(\overline{u})-\pi_*(u.\overline{u})$ is psef, as claimed. We now give a generalization of this to the case of a finite blowup and to meromorphic maps.

Proposition 2.2. *1) Let X be a compact Kähler manifold, and $\pi : Z\to X$ a finite composition of blowups along smooth centers. Further, let $u\in H^{1,1}(Z)$ be a $(1,1)$ cohomology class with complex coefficients. Then $(\pi)_*(u).(\pi)_*(\overline{u})-(\pi)_*(u.\overline{u})$ is a psef class.*

2) Let X and Y be compact Kähler manifolds, and $h : X\to Y$ a dominant meromorphic map. Further, let $u\in H^{1,1}(Y)$ be a cohomology class with complex coefficients on Y. Then $h^*(u).h^*(\overline{u})-h^*(u.\overline{u})$ is a psef class in $H^{2,2}(X)$.

Proof. 1) We prove by induction on the number of single blowups performed. If π is a single blowup then this follows from the above observation. Now assume that 1) is true when the number of single blowups performed is $\leq n$. We prove that 1) is true also when then number of single blowups performed is $\leq n+1$. We can decompose $\pi=\pi_1\circ\pi_2 : Z\to Y\to X$, where $\pi_2 : Z\to Y$ is a single blowup, and $\pi_1 : Y\to X$ is a composition of n single blowups. Apply the inductional assumption to π_1 and the cohomology class $(\pi_2)_*(u)$, we get

$$\pi_*(u).\pi_*(\overline{u})=(\pi_1)_*((\pi_2)_*(u)).(\pi_1)_*((\pi_2)_*(\overline{u}))\geq(\pi_1)_*((\pi_2)_*(u).(\pi_2)_*(\overline{u})).$$

Here the $\geq$ means that the difference of the two currents is psef. Now using the result for the single blowup π_2 and the fact that push-forward by the holomorphic map π_1 preserves psef classes, we have

$$(\pi_1)_*((\pi_2)_*(u).(\pi_2)_*(\overline{u}))\geq(\pi_1)_*(\pi_2)_*(u.\overline{u})=\pi_*(u.\overline{u}).$$

Hence $\pi_*(u).\pi_*(\overline{u})\geq\pi_*(u.\overline{u})$ as wanted.

2) By Hironaka's elimination of indeterminacies (see Hironaka [32] and Moishezon [35]), we can find a compact Kähler manifold Z, a finite blowup along smooth centers $\pi : Z\to X$ and a surjective holomorphic map $g : Z\to Y$ so that $h=g\circ\pi^{-1}$. By definition $h^*(u)=\pi_*g^*(u)$ and $h^*(u.\overline{u})=\pi_*(g^*(u.\overline{u}))=\pi_*(g^*(u).g^*(\overline{u}))$ (to see these equalities, we choose a smooth closed $(1,1)$ form α representing u and see immediately the equalities on the level

of currents). Therefore, apply 1) to the blowup $\pi : Z \to X$ and to the $(1,1)$ cohomology class $g^*(u)$ on Z, we obtain

$$h^*(u).h^*(\overline{u}) - h^*(u.\overline{u}) = \pi_*(g^*(u)).\pi_*(\overline{g^*(u)}) - \pi_*(g^*(u).\overline{g^*(u)}) \geq 0.$$

□

For the proofs of Theorems 1.1 and 1.2 we need to use the famous Hodge index theorem (Hodge-Riemann bilinear relations, see e.g. the last part of Chapter 0 in Griffiths-Harris [26]). Let X be a compact Kähler manifold of dimension k. Let $w \in H^{1,1}(X)$ be the cohomology class of a Kähler form on X. We define a Hermitian quadratic form which for cohomology classes with complex coefficients $u, v \in H^{1,1}(X)$ takes the value

$$\mathcal{H}(u,v) = u.\overline{v}.w^{k-2}.$$

Hodge index theorem says that the signature of $\mathcal{H}$ is $(1, h^{1,1}-1)$ where $h^{1,1}$ is the dimension of $H^{1,1}(X)$.

We are now ready for the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. First we show that there cannot be two non-collinear vectors $u_1, u_2 \in H^{1,1}(X)$ for which $f^*u_1 = \tau_1 u_1$ and $f^*u_2 = \tau_2 u_2$, where $\tau = \min\{|\tau_1|, |\tau_2|\} > \sqrt{\lambda_2(f)}$. Assume otherwise, we will show that for any u in the complex vector space of dimension 2 generated by u_1 and u_2, then $\mathcal{H}(u,u) \geq 0$ and this gives a contradiction to the Hodge index theorem. To this end, it suffices to show that $u.\overline{u}$ is psef. Let $u = a_1u_1 + a_2u_2$. For $n \in \mathbb{N}$, we define

$$v_n = \frac{a_1}{\tau_1^n}u_1 + \frac{a_2}{\tau_2^n}u_2.$$

Then it is easy to check that $(f^*)^n(v_n) = u$. Because f is 1-stable, we have from Proposition 2.2 that

$$u.\overline{u} = (f^*)^n(v_n).(f^*)^n(\overline{v_n}) = (f^n)^*(v_n).(f^n)^*(\overline{v_n}) \geq (f^n)^*(v_n.\overline{v_n}),$$

for any $n \in \mathbb{N}$. (Here the inequality $\geq$ means that the difference of the two cohomology classes is psef.) We fix an arbitrary norm $||\cdot||$ on the vector space $H^{1,1}(X)$. Then $||v_n||$ is bounded by $1/\tau^n$, hence the assumption that $\tau > \sqrt{\lambda_2(f)}$ implies that $(f^n)^*(v_n.\overline{v_n})$ converges to 0. Therefore, $u.\overline{u} \geq 0$ as wanted.

Hence $\lambda_1(f)$ is the unique eigenvalue of modulus $> \sqrt{\lambda_2(f)}$ of $f^* : H^{1,1}(X) \to H^{1,1}(X)$. It remains to show that $\lambda_1(f)$ is a simple root of the characteristic polynomial of $f^* : H^{1,1}(X) \to H^{1,1}(X)$. Assume otherwise, by using the Jordan normal form of a matrix, there will be two non-collinear vectors $u_1, u_2 \in H^{1,1}(X)$ for which $f^*(u_1) = \lambda_1(f)u_1$ and $f^*(u_2) = \lambda_1(f)u_2 + u_1$. Let $u = a_1u_1 + a_2u_2$. For any $n \in \mathbb{N}$ we define

$$v_n = \frac{a_1}{\lambda_1(f)^n}u_1 - \frac{na_2}{\lambda_1(f)^{n+1}}u_1 + \frac{a_2}{\lambda_1(f)^n}u_2.$$

Then it is easy to check that $(f^*)^n(v_n) = u$, and we can proceed as in the first part of the proof. □

Proof of Theorem 1.2. 1) First, we observe that for any $v \in H^{1,1}(X)$ with complex coefficients then $(f^*)^n(v).(f^*)^n(\overline{v}) \geq (f^*)^n(v.\overline{v})$ for all $n \in \mathbb{N}$. For example, we show how to do this for $n = 2$. Apply Proposition 2.2, we have

$$(f^*)^2(v).(f^*)^2(\overline{v}) = f^*(f^*(v)).f^*(\overline{f^*(v)}) \geq f^*(f^*(v).f^*(\overline{v})).$$

By Proposition 2.2 again and the assumption that $f^* : H^{2,2}(X) \to H^{2,2}(X)$ preserves psef classes, we obtain

$$f^*(f^*(v).f^*(\overline{v})) \geq (f^*)^2(v.\overline{v}),$$

and hence $(f^*)^2(v).(f^*)^2(\overline{v}) \geq (f^*)^2(v.\overline{v})$ as wanted.

We now finish the proof of 1). Let ω_X be a Kähler form on X. Then from the first part of the proof we get

$$(f^*)^n(\omega_X).(f^*)^n(\omega_X) \geq (f^*)^n(\omega_X^2),$$

for all $n \in \mathbb{N}$. For convenience, we let $||\cdot||$ denote an arbitrary norm on either $H^{1,1}(X)$ or $H^{2,2}(X)$. There is a constant $C > 0$ independent of n, so that for all $n \in \mathbb{N}$, we have

$$||(f^*)^n(\omega_X).(f^*)^n(\omega_X)|| \leq C||(f^*)^n(\omega_X)||^2 \leq C||(f^*)^n|_{H^{1,1}(X)}||^2,$$

and

$$C(f^*)^n(\omega_X^2) \geq ||(f^*)^n|_{H^{2,2}(X)}||.$$

(In the second inequality we used the assumption that $f^* : H^{2,2}(X) \to H^{2,2}(X)$ preserves the cone of psef classes.)

Therefore,

$$C^2||(f^*)^n|_{H^{1,1}(X)}||^2 \geq ||(f^*)^n|_{H^{2,2}(X)}||$$

for any $n \in \mathbb{N}$. Taking n-th root and letting $n \to \infty$, we obtain $r_1(f)^2 \geq r_2(f)$.

2) Using the ideas from the proofs Theorem 1.1 and 1), we obtain 2) immediately.

□

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Department of mathematics, Syracuse University, Syracuse NY 13244, USA
E-mail address: tutruong@syr.edu