DEGREE COMPLEXITY OF A FAMILY OF BIRATIONAL MAPS: II. EXCEPTIONAL CASES

TUYEN TRUNG TRUONG

Abstract. We determine the degree complexity for all elements of a family k_F of birational maps which was introduced and studied in [7].

1. Introduction

Let $\mathbf{P}^2$ denote the complex projective space, and let $f : \mathbf{P}^2 \to \mathbf{P}^2$ be a rational map. We will consider its iterates $f^n = f \circ f \circ \cdots \circ f$. A basic invariant of iteration is the degree complexity, or the exponential rate of growth:

$$\delta(f) = \lim_{n\to\infty} (deg(f^n))^{1/n}. \tag{1.1}$$

Here we consider the family of birational maps k_F defined in Section 2 below for an arbitrary polynomial F. If we regard

$$F_a(w) = a_0 + a_1 w + \cdots + a_N w^N \tag{1.2}$$

as depending on the complex parameters $a = (a_0, \ldots, a_N) \in \mathbb{C}^{N+1}$, then the dependence $a \mapsto \delta(k_{F_a})$ is lower semi-continuous in the Zariski topology. This means that the set $\{a : \delta(k_{F_a}) \leq t\}$ is an algebraic variety for all t. In particular, the value of $\delta(k_{F_a})$ is equal to a constant value δ^*_N outside a proper subvariety of $\mathbb{C}^{N+1}$.

A parameter a is said to be exceptional if $\delta(k_{F_a}) < \delta^*_N$. Exceptional maps are of special interest because the lower degree growth indicates the presence of internal symmetries and non-generic behaviors. Such symmetries often make δ more difficult to compute. For instance, there is a birational map K on the projectivized space of $q \times q$ matrices (see [8], and [11]). The degree growth of the restriction of the map K to the space of cyclic matrices was shown to be the largest root of the polynomial $x^2 - (q^2 - 4q + 2)x + 1$ (see [9]). However, the degree growth of the same map K, restricted to the smaller space of cyclic, symmetric matrices, depends in a much more complicated way on the number q (for primes q it was determined in [4], and for general q it was determined in [6]).

In the case of the family k_{F_a}, the numbers δ^*_N were determined in [7]. Here we consider the map $a \to \delta_{F_a}$ for the full family; we determine the exceptional values as well as the associated rates of degree growth.

Theorem 1. *Suppose that F_a is as above, and $N = deg(F_a)$ is even. If $a_0 = 2/(m+1)$ for some integer $m \geq 0$, then $\delta(k_{F_a})$ is the largest root of the polynomial $x^{2m+1}(x^2 - (N+1)x - 1) + x^2 + N$. Otherwise, $\delta(k_{F_a}) = \delta^*_N$ is the largest root of $x^2 - (N+1)x - 1$.*

Date: January 22, 2009.
2000 *Mathematics Subject Classification.* 37F10.
Key words and phrases. Birational maps, Degree complexity.

The behavior of this family is more complicated when the degree N is odd. For instance, we have

Theorem 2. *Suppose that F_a is as above, and $N = 3$. Then we have the following cases:*

Case 1: $a_2 \neq a_3$.

If $a_0 = 2/(1 + m)$ for some integer $m \geq 0$, then $\delta(k_F)$ is the largest root of the polynomial $x^{2m+1}(x^3 - 3x^2 - 4x - 1) + x^3 + x^2 + 3x + 2$. Otherwise, $\delta(k_F) = \delta_3^*$ is the largest root of the polynomial $x^3 - 3x^2 - 4x - 1$.

Case 2: $a_2 = a_3$.

2a. If $a_0 = 2$, then k_F is an automorphism, $\delta(k_F) = 1$. Moreover the degree growth is quadratic.

2b. If $a_0 = 2/(1 + m)$ for some integer $m \geq 1$, then $\delta(k_F)$ is the largest root of the polynomial $x^{2m}(x^3 - 3x^2 - 2x - 1) + x^2 + x + 3$.

2c. If $a_0 = 2 + \frac{l}{2(1+l)}$ for some integer $l \geq 1$, then $\delta(k_F)$ is the largest root of the polynomial $x^{2l+2}(x^3 - 3x^2 - 2x - 1) + 3x^2 + x + 1$.

2d. Otherwise, $\delta(k_F)$ is the largest root of the polynomial $x^3 - 3x^2 - 2x - 1$.

So we see that for case $N = 3$, there are (infinitely many) linear functions L_1, L_2, $L_3, \ldots$ depending on the variable $a = (a_0, a_1, a_2, a_3)$, and the different cases are determined by conditions of the form $L_s(a) = 0$ for certain values of s, and $L_t(a) \neq 0$ for certain values of t. Thus the sets of exceptional parameters are constructed by linear functions.

We will find in Section 5 that this is typical of the general case for N odd. We also find that there are no automorphisms in the family k_{F_a} other than the ones given in [7].

One difference between the cases when degree N is even or odd is the following. When N is even, the exceptional cases are characterized by a single condition whether $a_0 = 2/(1 + m)$ for some integer $m \geq 0$ or not. When N is odd there is in addition other conditions for exceptional cases, the number of these exceptional conditions are $(N + 3)/2$. In the proofs of Theorems 1 and 3, following the general frame of Diller and Favre in [12] for working with birational maps of a surface, we will construct spaces Z which is a composition of finite point-blowups of $\mathbb{P}^2$, whose induce map k_Z is good (say, A.S. or 1-regular, see [13] for details). We mention here a special phenomena that happens when N is odd: if j exceptional conditions are satisfied, we need to construct spaces $Z_1, \ldots, Z_j$ where each Z_{l+1} is a composition of two point-blowups of Z_l. In other words, if N is odd, when a new exceptional condition occurs, we need to blowups two more points.

Acknowledgment The author would like to thank Professor Bedford for his many helpful suggestions. We also would like to thank the referee for her/his helpful comments, in particular for suggesting the proof of Theorem 4 based on Lemma 4 that we include in this paper.

2. Properties of k_F

With F as in (1.2), we define two involutions:

$$j_F(x, y) = (-x + F(y), y),\ i(x, y) = (1 - x - \frac{x - 1}{y}, -y - 1 - \frac{y}{x - 1}).$$

and we set $k = k_F = j_F \circ i$.

We recall the following sets from [7]:

$$\begin{aligned}
C_1 &= \{x_0 = 0\},\ C_2 = \{x_0 = x_1\},\ C_3 = \{x_2 = 0\}, C_4 = \{-x_0^2 + x_0x_1 + x_1x_2 = 0\},\\
C_1' &= C_1,\ C_2' = \{1 + \frac{x_1}{x_0} - F(\frac{x_2}{x_0}) = 0\},\ C_3' = C_3,\\
C_4' &= \{\frac{x_2}{x_0} - (1 + \frac{x_2}{x_0})(1 + \frac{x_1}{x_0} - F(\frac{x_2}{x_0})) = 0\}.
\end{aligned}$$

The exceptional hypersurfaces of k_F are mapped as

$$k_F:\ C_4 \mapsto [1 : -1 + a_0 : 0] \in C_3,\ C_1 \cup C_2 \cup C_3 \mapsto e_1.$$

The points of indeterminacy of k_F are $e_1 = [0 : 1 : 0], e_2 = [0 : 0 : 1]$, and $e_{01} = [1 : 1 : 0]$. The exceptional curves for k_F^{-1} are mapped as

$$k_F^{-1}:\ C_1' \cup C_3' \mapsto e_1,\ C_2' \mapsto e_2,\ C_4' \mapsto e_{01}.$$

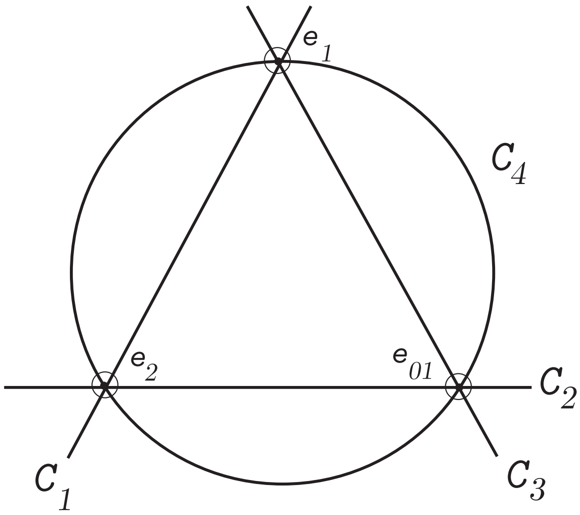

FIGURE 1. Indeterminacy points and exceptional curves of k_F.

3. DEGREE N IS EVEN

Let us start by recalling the space X constructed in Section 3 of [7]. We define the complex manifold $\pi_X : X \to \mathbb{P}^2$ (see Figure 3.1 in [7]) by blowing up points $e_1, p_1, \ldots, p_{N-1}$ in the following order:

i) blowup $e_1 = [0 : 1 : 0]$ and let E_1 denote the exceptional fiber over e_1,

ii) blowup $q = E_1 \cap C_4$ and let Q denote the exceptional fiber over q,

iii) blowup $p_1 = E_1 \cap C_1$ and let P_1 denote the exceptional fiber over e_1,

iv) blowup $p_j = P_{j-1} \cap E_1$ with exceptional fiber P_j for $2 \le j \le N - 1$.

Here we use the notational convention that if S is a curve at one stage of the construction, then S will denote its strict transforms at subsequent stages.

The coordinate projection at P_j $(1 \le j \le N - 1)$ is chosen as follows

$$\pi_j : X \ni (s, u) \mapsto [s^{j+1}u : 1 : s^j u] \in \mathbb{P}^2.$$

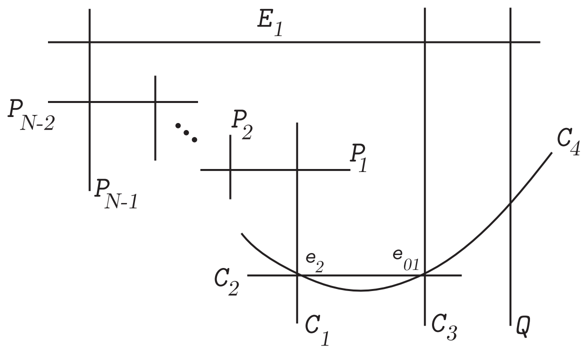

FIGURE 2. The space X. All exceptional fibers are lying over e_1.

In this coordinate $P_j = \{s = 0\}$. For convenience we will use the notations $u \in P_j$ or $[u]_{P_j}$ to indicate the point of P_j which has coordinate $(0, u)$ in this coordinate projection.

Let $k_X := \pi_X^{-1} \circ k_F \circ \pi_X$ denote the induced birational map of X. The exceptional curves for k_X are $C_1, C_2, C_4, P_1, \ldots, P_{N-2}$. The curves $C_1, C_2, P_1 \ldots, P_{N-2}$ are mapped to the same point $1/a_N \in P_{N-1}$, while C_4 is mapped to the point $[1 : -1 + a_0 : 0] \in C_3$. By Lemmas 3.2 and 3.3 in [7], the only way that an exceptional curve can be mapped to a point of indeterminacy is that $a_0 = 2/(m+1)$ for some integer $m \geq 0$, and in this case we have $k_X^{2m+1} C_4 = [1 : 1 : 0]$.

If $a_0 = 2/(m+1)$ we construct the new manifold Z by blowing up the manifold X at the points

$$\begin{aligned}
r_0 &= k_X(C_4) = [1 : -1 + a_0 : 0] \in C_3, \\
q_1 &= k_X(r_0) \in Q,\ r_1 = k_X(q_1) \in C_3, \\
\ldots & \\
q_m &= k_X(r_{m-1}) \in Q,\ r_m = k_X(q_m) = [1 : 1 : 0] = e_{01} \in C_3.
\end{aligned}$$

Call $R_0, Q_1, R_1, \ldots, Q_m, R_m$ the exceptional fibers of this blowup. Let k_Z be the induced birational map of Z.

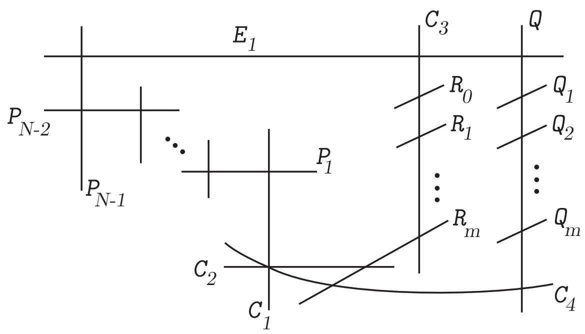

FIGURE 3. The space Z when $a_0 = 2/(m+1)$. New exceptional fibers are lying on $e_{01} = r_m$ and its pre-images.

Lemma 1. *If* $a_0 = 2/(m+1)$ *then the curves* $C_4, R_0, Q_1, R_1, \ldots, Q_m, R_m$ *are not exceptional for* k_Z.

Proof. It suffices to check that C_4 is not exceptional. We choose a local projection for R_0 as

$$Z \ni (s,u) \mapsto [1 : -1 + a_0 + su : s].$$

In this coordinate chart $R_0 = \{s = 0\}$. If we rewrite $k[x_0 : x_1 : x_2]$ as

$$k[x_0 : x_1 : x_2] = [1 : -1 - \frac{x_0^2 - x_0x_1 - x_1x_2}{x_0x_1} + F(\frac{x_0^2 - x_0x_1 - x_1x_2}{x_0(x_1 - x_0)}) : \frac{x_0^2 - x_0x_1 - x_1x_2}{x_0(x_1 - x_0)}]$$

then it can be seen that

$$k_Z : C_4 \ni [x_0 : x_1 : x_2] \mapsto a_1 + \frac{x_1}{x_0} \in R_0.$$

Hence C_4 is not exceptional. □

The induced map k_Z acts as follows

$$k_Z : E_1 \mapsto E_1,\ P_{N-1} \mapsto P_{N-1},\ C_1, C_2, P_1, \ldots, P_{N-2} \mapsto \frac{1}{a_N} \in P_{N-1},\ Q \mapsto C_3 \mapsto Q,$$
$$k_Z : C_4 \mapsto R_0 \mapsto Q_1 \mapsto R_1 \mapsto \ldots \mapsto Q_m \mapsto R_m \mapsto C_4',$$
$$k_Z^{-1} : C_1, P_1, \ldots, P_{N-1} \mapsto -\frac{1}{a_N} \in P_{N-1}.$$

If S is a curve in Z, we will use the notation S to denote its class in $Pic(Z)$. Let $H \in Pic(Z)$ denote the class of a generic line. Then $H, E_1, P_1, \ldots, P_{N-1}, Q, Q_1, \ldots, Q_m, R_0, \ldots, R_m$ form an ordered basis for the space $Pic(Z)$. The curves C_1, C_2, C_3, C_4 can be represented in this basis as

$$\begin{aligned}
C_1 &= H - E_1 - Q - \sum_{j=1}^{N-1} (j+1)P_j - \sum_{j=1}^{m} Q_j,\\
C_2 &= H - R_m,\\
C_3 &= H - E_1 - Q - \sum_{j=1}^{N-1} jP_j - \sum_{j=1}^{m} Q_j - \sum_{j=0}^{m} R_j,\\
C_4 &= 2H - E_1 - 2Q - \sum_{j=1}^{N-1} jP_j - 2\sum_{j=1}^{m} Q_j - R_m.
\end{aligned}$$

From this, we see that $k_Z^*: Pic(Z)\rightarrow Pic(Z)$ is as follows

$$\begin{aligned}
k_Z^*(H) &= (2N+1)H-NE_1-(N+1)Q-(N+1)\sum_{j=1}^{N-1}jP_j-(N+1)\sum_{j=1}^{m}Q_j-(N+1)R_m,\\
k_Z^*(E_1) &= E_1,\\
k_Z^*(Q) &= C_3=H-E_1-Q-\sum_{j=1}^{N-1}jP_j-\sum_{j=1}^{m}Q_j-\sum_{j=0}^{m}R_j,\\
k_Z^*(P_j) &= 0,\ 1\leq j\leq N-2,\\
k_Z^*(P_{N-1}) &= C_1+C_2+\sum_{j=1}^{N-1}P_j=2H-E_1-Q-\sum_{j=1}^{N-1}jP_j-\sum_{j=1}^{m}Q_j-R_m,\\
k_Z^*(R_0) &= C_4=2H-E_1-2Q-\sum_{j=1}^{N-1}jP_j-2\sum_{j=1}^{m}Q_j-R_m,\\
k_Z^*(R_j) &= Q_j,\ 1\leq j\leq m,\ k_Z^*(Q_j)=R_{j-1},\ 1\leq j\leq m.
\end{aligned}$$

Proof of Theorem 1: If $a_0\neq 2/(1+m)$ for any integer $m\geq 0$, $\delta(k_F)$ was computed in [7]. It was shown in this case that $\delta(k_F)$ is the largest root of $x^2-(N+1)x-1$.

Let us suppose now that $a_0=2/(1+m)$ for some integer $m\geq 0$. Then by Lemma 1, we see that for every exceptional curve Γ, the images $k_Z^j(\Gamma)$, $j\geq 1$, are disjoint from the determinacy locus. It follows that $(k_Z^n)^*=(k_Z^*)^n$ for all integer $n\geq 1$. It follows that $\delta(k_F)$ is the spectral radius of k_Z^*. Thus it is the largest root of the characteristic polynomial of k_Z^*, which is

$$P(x)=-x[x^{2m+1}(x^2-(N+1)x-1)+x^2+N].$$

4. Degree $N=3$

In this section we will prove Theorem 2. First we consider the more general case where $N\geq 3$ is an odd number. First we recall the construction of spaces Y and Y_1 constructed in [7]. We start from the space X constructed in the previous section. Then the line C_1 and all blowup fibers P_j $(1\leq j\leq N-2)$ are all exceptional for both k_X and k_X^{-1}. Moreover C_2 is exceptional for k_X:

$$\begin{aligned}
k_X:\ & C_1,C_2,P_1,\ldots,P_{N-2}\mapsto\frac{1}{a_N}\in P_{N-1},\\
k_X^{-1}:\ & C_1,P_1,\ldots,P_{N-2}\mapsto\frac{1}{a_N}\in P_{N-1}.
\end{aligned}$$

Hence when N is odd the image of all exceptional curves of k_X concide with a point of indeterminacy $\zeta_0=\frac{1}{a_N}\in P_{N-1}$. Let $\pi_Y:Y\rightarrow\mathbb{P}^2$ be the blowup of X at the point $\zeta_0\in P_{N-1}$, and let P_N be the exceptional fiber. At P_N we use the coordinate projection

$$\pi_N:(u,s)\in Y\mapsto[s^N(su+\zeta_0):1:s^{N-1}(su+\zeta_0)]\in\mathbb{P}^2.$$

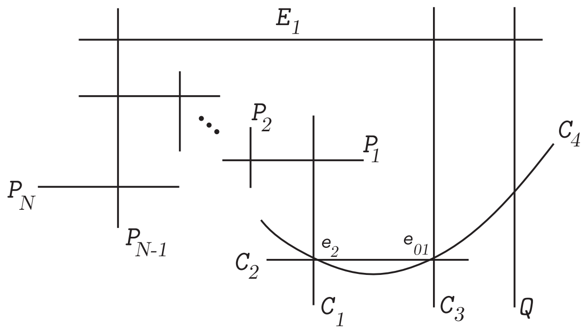

FIGURE 4. The space Y which is the blowup of X at a point on P_{N-1}.

We set $Y_1 = Y$ if $a_0 \neq \frac{2}{m+1}$ for every integer $m \geq 0$. Otherwise, as in the previous section, define $\pi_1 : Y_1 \to \mathbb{P}^2$ to be the blowup of Y at the points

$$\begin{aligned} r_0 &= [1 : -1 + a_0 : 0] \in C_3, \\ q_1 &= k_X(r_0) \in Q,\ r_1 = k_X(q_1) \in C_3, \\ \cdots & \\ q_m &= k_X(r_{m-1}) \in Q,\ r_m = k_X(q_m) = [1 : 1 : 0] \in C_3, \end{aligned}$$

and call $R_0, Q_1, R_1, \ldots, Q_m, R_m$ the exceptional fibers of this blowup.

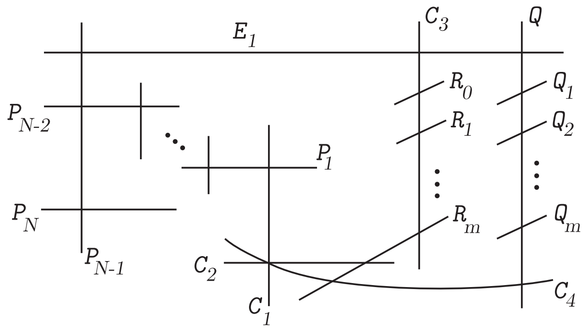

FIGURE 5. The space Y_1 in case $a_0 = 2/(1 + m)$.

Lemma 2. *k_{Y_1} maps $P_N \longleftrightarrow P_{N-2}$ by the following formulas*

$$P_N \ni u \mapsto \frac{1}{-a_N^2 u - (N-1)a_N + a_{N-1}} \in P_{N-2},$$

$$P_{N-2} \ni u \mapsto -\frac{1 + a_{N-1}u}{a_N^2 u} \in P_N.$$

Proof. (Sketch) First, write

$$k[s^N(su + \zeta_0) : 1 : s^{N-1}(su + \zeta_0)] = [z_0(s, u) : z_1(s, u) : z_2(s, u)],$$

then if $u\in P_N$ its image $w\in P_{N-2}$ under k_{Y_1} can be computed as

$$w=\lim_{s\rightarrow 0}\frac{z_2^{N-1}}{z_0^{N-2}z_1}.$$

Now to compute the image of $w\in P_{N-2}$ we do as follows: If $u\in P_N$ then applying the above argument to the inverse map k^{-1} will give its image $w=g(u)\in P_{N-2}$ under the map $k_{Y_1}^{-1}$. Now the inverse $u=g^{-1}(w)$ is the image of $w\in P_{N-2}$ under the map k_{Y_1}. □

In a similar way, we also have

Lemma 3. *If we set*

$$\begin{aligned}\zeta_1&=-\frac{a_{N-1}}{a_N^2},\\ \xi_1&=\frac{-(N-1)a_N+a_{N-1}}{a_N^2},\end{aligned}$$

then

$$\begin{aligned}&k_{Y_1}:C_1,C_2,P_1,P_2,\dots,P_{N-3}\mapsto\zeta_1\in P_N,\\ &k_{Y_1}^{-1}:C_1,P_1,P_2,\dots,P_{N-3}\mapsto\xi_1\in P_N.\end{aligned}$$

Hence the map $k_{Y_1}^2:P_N\rightarrow P_N$ is

$$P_N\ni u\mapsto u+\frac{(N-1)a_N-2a_{N-1}}{a_N^2}=u+\zeta_1-\xi_1\in P_N. \tag{4.1}$$

From (4.1), we see that the orbit of ζ_1 (hence also the orbit of all exceptional curves of k_{Y_1}) is

$$k_{Y_1}^{2m}(\zeta_1)=\zeta_1+m(\zeta_1-\xi_1)\in P_N. \tag{4.2}$$

Hence the orbit of all exceptional curves of k_{Y_1} will contain a point of indeterminacy iff that indeterminacy point is ξ_1, that is iff $\zeta_1+m(\zeta_1-\xi_1)=\xi_1$. The last condition is satisfied iff $\xi_1=\zeta_1$, that is iff the coefficients a_N and a_{N-1} of the polynomial $F_N(z)$ satisfy the linear equation

$$-a_{N-1}=-(N-1)a_N+a_{N-1}.$$

Hence if $a_{N-1}\neq\frac{(N-1)a_N}{2}$ then the map k_{Y_1} satisfies the condition $(k_{Y_1}^n)^*=(k_{Y_1}^*)^n$ for all integer $n\geq 1$, while if $a_{N-1}=\frac{N-1)a_N}{2}$ then the image of all exceptional curves of k_{Y_1} is the point of indeterminacy $\zeta_1=\xi_1$.

Proof of Theorem 2: Let Y_1 be as above. Since $N=3$ we have

$$\begin{aligned}\zeta_1&=-\frac{a_2}{a_3^2},\\ \xi_1&=\frac{-2a_3+a_2}{a_3^2}.\end{aligned}$$

Then (4.2) becomes

$$k_Y^{2m}(-\frac{a_2}{a_3^2})=-\frac{a_2}{a_3^2}-m\frac{2a_2-2a_3}{a_3^2}. \tag{4.3}$$

In Case 1: $a_2\neq a_3$, it follows that the orbit of exceptional curves of k_{Y_1} does not contain a point of indeterminacy. Thus $(k_{Y_1}^n)^*=(k_{Y_1}^*)^n$ for all integer $n\geq 1$, and so $\delta(k_F)$ is the spectral radius of k_Y^*, which is the largest root of the polynomial given in the statement of Theorem 2.

In Case 2: $a_2 = a_3$, we have $\zeta_1 = -\xi_1 = -\frac{1}{a_3}$. Hence ζ_1 and ξ_1 are both the image of exceptional curves C_1, C_2 of k_{Y_1}, and the image of the exceptional curve C_1 of $k_{Y_1}^{-1}$. We define a complex manifold $\pi_{Y_2} : Y_2 \to \mathbb{P}^2$ by blowing up Y_1 at the point $-\frac{1}{a_3} \in P_3$, and call P_4 the exceptional fiber of this blowup. We use a local coordinate projection at P_4 as follows:

$$\pi_4 : \ Y_2 \ni (s,u) \mapsto [s^3(s^2u - \frac{1}{a_3}s + \frac{1}{a_3}) : 1 : s^2(s^2u - \frac{1}{a_3}s + \frac{1}{a_3})] \in \mathbb{P}^2.$$

The induced map k_{Y_2} is as follows:

$$\begin{aligned} k_{Y_2} &: P_4 \ni u \mapsto [0 : 1 : \frac{1}{-a_3 + a_1 + a_3^2 u}] \in C_1, \\ k_{Y_2} &: C_1 \ni [0:1:u] \mapsto \frac{1 + (a_3 - a_1)u}{a_3 u} \in P_4, \\ k_{Y_2} &: C_2 \mapsto [\frac{a_3 - a_1}{a_3^2}]_{P_4} \mapsto [0:0:1] = e_2. \end{aligned}$$

Thus the orbit of the exceptional curve C_2 encounters an indeterminacy point.

Let $\pi_{Y_3} : Y_3 \to \mathbb{P}^2$ be the complex manifold obtained by blowing up Y_2 at two points $e_2 = [0:0:1]$ and $\frac{a_3 - a_1}{a_3^2} \in P_4$, and let E_2 and P_5 be the exceptional fibers of this blowup. We use a local coordinate projection at P_5 as

$$Y_3 \ni (s,u) \mapsto [s^3(s^3u + s^2\frac{a_3 - a_1}{a_3^2} - s\frac{1}{a_3} + \frac{1}{a_3}) : 1 : s^2(s^3u + s^2\frac{a_3 - a_1}{a_3^2} - s\frac{1}{a_3} + \frac{1}{a_3})],$$

and use a local coordinate projection at E_2 as

$$E_2 \ni (s,u) \mapsto [s : su : 1] \in \mathbb{P}^2.$$

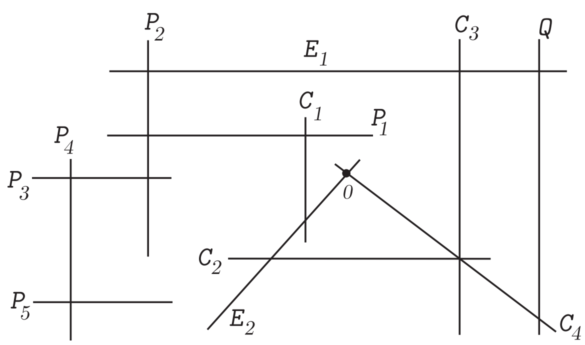

FIGURE 6. The space Y_3 in case $N = 3$ and $a_2 = a_3$.

Then the induced map k_{Y_3} is as follows:

$$k_{Y_3}: P_5 \ni u \mapsto -a_3^2 u - a_3 + 2a_1 + a_0 - 4 \in E_2,$$
$$k_{Y_3}: E_2 \ni u \mapsto \frac{-u - a_3 + 2a_1 - a_0 + 1}{a_3^2} \in P_5,$$
$$k_{Y_3}^2: E_2 \ni u \mapsto u + 2a_0 - 5 \in E_2,$$
$$k_{Y_3}: C_2 \mapsto [-\frac{a_3 - 2a_1 + a_0}{a_3^2}]_{P_5} \mapsto 2a_0 - 4 \in E_2.$$

Hence the orbit of the point $2a_0 - 4 \in E_2$ is

$$k_{Y_3}^{2l}(2a_0 - 4) = 2a_0 - 4 + l(2a_0 - 5),\ l \geq 0. \quad (4.4)$$

The point $0 \in E_2$ is a point of indeterminacy for k_{Y_3}.

2a. If $a_0 \neq 2 + \frac{l}{2(l+1)}$ for $l \geq 0$, then from (4.4), the orbit of the exceptional curve C_2 of k_{Y_3} does not contain the point of indeterminacy $0 \in E_2$. It follows $(k_{Y_3}^n)^* = (k_{Y_3}^*)^n$ for all integer $n \geq 1$. Then a computation of $k_{Y_3}^*$ on $H^{1,1}(Y_3)$ similar to that of Section 3 completes the proof of Theorem 4 for this case.

2b. If $a_0 = 2 + \frac{l}{2(l+1)}$ for an integer $l \geq 0$, then from (4.4) it follows that the orbit of C_2 contains the point of indeterminacy $0 \in E_2$. We define a complex manifold $\pi_Z : Z \to \mathbb{P}^2$ by blowing up Y_3 at the points

$$p_6 = [-\frac{a_3 - 2a_1 + a_0}{a_3^2}]_{P_5},\ s_0 = k_{Y_3}(s_0) = [2a_0 - 4]_{E_2},$$
$$s_1 = k_{Y_3}^2(s_0), \ldots, s_{2l} = k_{Y_3}^{2l+1}(s_0) = [0]_{E_2},$$

and let $P_6,\ S_0, S_1, \ldots, S_{2l}$ the exceptional fibers of this blowup. Then, as in the proof of Lemma 1, it can be shown that the curves $C_2,\ P_6, S_0, \ldots, S_{2l}$ are not exceptional for k_Z. It follows $(k_Z^n)^* = (k_Z^*)^n$ for all integer $n \geq 1$. Then a computation of k_Z^* on $H^{1,1}(Z)$ similar to that of Section 3 completes the proof of Theorem 2 for this case.

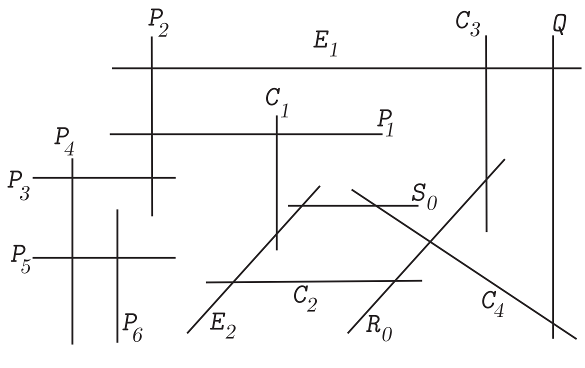

FIGURE 7. The space Z in case $N = 3$, $a_2 = a_3$, and $a_0 = 2$.

5. DEGREE N IS ODD

In this section we will describe the degree complexities of all elements of the family k_F having odd degrees.

For fixed N, define for $0 \le j \le N$

$$L_j(a_0, a_1, \dots, a_N) = (a_{N-j} + a_{N-j+1}) - \sum_{l=0}^{j} (-1)^l a_{N-l} \binom{N-l}{j-l}, \tag{5.1}$$

where $\binom{n}{j}$ is the binomial coefficient.

These linear functions will determine all exceptional parameters of the family k_F when $deg(F) = N$ is odd.

Theorem 3. *Suppose that $N = deg(F) \ge 3$ is odd. Define h as the largest integer in $[0, N-2]$ for which*

$$L_j(a_0, a_1, \dots, a_n) = 0$$

for all $0 \le j \le h$. Then exactly one of the following occurs:

Case 1: $h < N-2$.

If $a_0 = 2/(1+m)$ for some integer $m \ge 0$, then $\delta(k_F)$ is the largest real root of the polynomial $(1+x^{2m+1})[x^3 - Nx^2 - (N-h+1)x - 1] + (N+1)x^2 + (2N-h+1)x + N - h$. Otherwise $\delta(k_F)$ is the largest real root of the polynomial $x^3 - Nx^2 - (N+1-h)x - 1$.

Case 2: $h = N-2$.

2a. If $a_0 = 2/(1+m)$ for some integer $m \ge 0$, and $a_0 = \frac{N+1}{2} + \frac{l}{2(1+l)}$ for some integer $l \ge 0$, then $N = 3$, $a_0 = 2$, and the map k_F is an automorphism with $\delta(k_F) = 1$. Moreover the degree growth is quadratic.

In the remaining cases, we assume that $N \ge 5$.

2b. If $a_0 = 2/(1+m)$ for some integer $m \ge 0$, then $\delta(k_F)$ is the largest real root of the polynomial $x^{2m}(x^3 - Nx^2 - 2x - 1) + x^2 + x + N$.

2c. If $a_0 = \frac{N+1}{2} + \frac{l}{2(1+l)}$ for some integer $l \ge 0$, then $\delta(k_F)$ is the largest real root of the polynomial $x^{2l+2}(x^3 - Nx^2 - 2x - 1) + Nx^2 + x + 1$.

2d. Otherwise, $\delta(k_F)$ is the largest real root of the polynomial $x^3 - Nx^2 - 2x - 1$.

The proof of this Theorem will be given in Section 7, but here we discuss how the linear functions L_j are derived.

Since $F_N(z)$ is a polynomial of degree N, the function

$$s^N F\left(-1 - \frac{1}{s}\right) + (1+s)s^N F\left(\frac{1}{s}\right)$$

is a polynomial of degree $N+1$, and we have

$$s^N F\left(-1 - \frac{1}{s}\right) + (1+s)s^N F\left(\frac{1}{s}\right) = a_0 s^{N+1} + \sum_{j=0}^{N} L_j(a_0, \dots, a_N) s^j.$$

The numbers ζ_1 and ξ_1 in the previous section can be constructed as follows:

$$\begin{aligned} \frac{1}{a_N^2 s}\left[\frac{1+s}{\zeta_0 + su} - (1+s)s^N F\left(\frac{1}{s}\right)\right] &= \zeta_1 - u + O(s), \\ \frac{1}{a_N^2 s}\left[\frac{1+s}{\zeta_0 + su} + s^N F\left(-1 - \frac{1}{s}\right)\right] &= \xi_1 - u + O(s). \end{aligned}$$

Then

$$\zeta_1 - \xi_1 = -\frac{1}{a_N^2 s}\left[s^N F\left(-1 - \frac{1}{s}\right) + (1+s)s^N F\left(\frac{1}{s}\right)\right] + O(s).$$

Hence $\zeta_1 - \xi_1 = -L_1(a_0, \dots, a_N)/a_N^2$, so the vanishing of L_1 corresponds to the case $\zeta_1 = \xi_1$.

If $\zeta_1=\xi_1$ and $N\geq 5$, define complex numbers ζ_2 and ξ_2 as follows

$$\begin{aligned}\frac{1}{a_N^2s^2}\left[\frac{1+s}{\zeta_0+s\zeta_1+s^2u}-(1+s)s^NF\left(\frac{1}{s}\right)\right] &= \zeta_2-u+O(s),\\ \frac{1}{a_N^2s^2}\left[\frac{1+s}{\zeta_0+\zeta_1s+s^2u}+s^NF\left(-1-\frac{1}{s}\right)\right] &= \xi_2-u+O(s).\end{aligned}$$

Then $\zeta_2-\xi_2=-L_2(a_0,\ldots,a_N)/a_N^2$ (ζ_2 and ξ_2 will play the similar roles to that of ζ_1 and ξ_1). However $L_2=nL_1/2=0$ hence $\zeta_2=\xi_2$. Then, if we define ζ_3 and ξ_3 by

$$\begin{aligned}\frac{1}{a_N^2s^3}\left[\frac{1+s}{\zeta_0+\zeta_1s+\zeta_2s^2+s^3u}-(1+s)s^NF\left(\frac{1}{s}\right)\right] &= \zeta_3-u+O(s),\\ \frac{1}{a_N^2s^3}\left[\frac{1+s}{\zeta_0+\zeta_1s+\zeta_2s^2+s^3u}+s^NF\left(-1-\frac{1}{s}\right)\right] &= \xi_3-u+O(s).\end{aligned}$$

we have $\zeta_3-\xi_3=-L_3(a_0,\ldots,a_N)/a_N^2$. Note that now L_3 is not a linear combination of L_1 and L_2, hence in general $\zeta_3\neq\xi_3$.

Continuing, we assume that $\zeta_1=\xi_1$, $\zeta_2=\xi_2$, $\zeta_3=\xi_3$ and $N\geq 7$. Then we can define ζ_4 and ξ_4 in the same manner, and $\zeta_4-\xi_4=-L_4(a_0,\ldots,a_N)/a_N^2$. Note in this case that L_4 is a linear combination of L_1, L_2 and L_3. Hence $\zeta_4=\xi_4$. We can continue defining ζ_5 and ξ_5, and have that $\zeta_5-\xi_5=-L_5(a_0,\ldots,a_N)/a_N^2$. Moreover L_5 is not a linear combination of L_1, L_2, L_3 and L_4.

In Section 6 we will show that L_{2j} is a linear combination of $L_1,\ldots,L_{2j-1}$, while L_{2j+1} is not a linear combination of $L_1,\ldots,L_{2j}$ for all j.

As a consequence of Theorem 3, we show that there are no automorphisms in the family k_F other than the ones given in [7]. The following lemma, which is used in the proof of Theorem 4, and also the proof of Theorem 4, are suggested to us by the referee.

Lemma 4. *a) Let $f:\mathcal{Z}\rightarrow\mathcal{Z}$ be an automorphism of a surface $\mathcal{Z}$. Let $\delta(f)$ be the spectral radius of the linear map (which is also the complexity degree of f) $f^*:H^{1,1}(\mathcal{Z})\rightarrow H^{1,1}$. Then either $\delta(f)=1$, or the minimal polynomial $p(x)$ of $\delta(f)$ is symmetric. That is, if d is the degree of $p(x)$ we have*

$$x^dp(1/x)=p(x).$$

b) Let $f:\mathcal{Z}\rightarrow\mathcal{Z}$ be a birational map of a surface $\mathcal{Z}$. Assume that f is 1-regular and that f is birational equivalent to an automorphism. Let $\chi(f)(x)$ be the characteristic polynomial of the linear map $f^*:H^{1,1}(\mathcal{Z})\rightarrow H^{1,1}(\mathcal{Z})$. Then roots of $\chi(f)(x)$ are $\delta(f)$, $1/\delta(f)$, and/or 0, and/or algebraic numbers of complex modulus 1.

In particular, let $g(x)$ be a factor of $\chi(f)(x)$ which is a monic polynomial and whose coefficients are integers and such that $g(0)\neq 0$. Then g is either symmetric or anti-symmetric. That is if d is the degree of $g(x)$ we have:

either

$$x^dg(1/x)=g(x),$$

or

$$x^dg(1/x)=-g(x).$$

Moreover if g is anti-symmetric then $g(1)=0$.

Proof. We remark first that if $f:\mathcal{Z}\rightarrow\mathcal{Z}$ is a birational map of a surface $\mathcal{Z}$, then $\chi(f)(x)$ is a monic polynomial (that is it is a polynomial with integer coefficients, and its leading coefficient is 1). Hence if $\lambda\in\mathbb{Q}$ is a root of $\chi(f)(x)$ then either $\lambda\in\mathbb{N}$ or 0.

a) Since $f:\mathcal{Z}\rightarrow\mathcal{Z}$ is an automorphism, both f and f^{-1} are 1-regular. Moreover $(f^{-1})^*=(f^*)^{-1}$ and $\delta(f)=\delta(f^{-1})$. By $(f^{-1})^*=(f^*)^{-1}$ we have that if λ is any root of $\chi(f)$ then $1/\lambda$ is a root of $\chi(f^{-1})$, and vice versa. In particular, $1/\delta(f)$ is also a root of $\chi(f)$.

We claim that roots of $\chi(f)$ are $\delta(f)$, $1/\delta(f)$, and/or algebraic numbers of complex norm 1. We have two cases:

-Case 1: $\delta(f)=1$. Then $\chi(f)$ can not have roots λ with $|\lambda|<1$. Because otherwise then $1/\lambda$ is a root of $\chi(f^{-1})$ with $|1/\lambda|>1$, which is a contradiction to $\delta(f^{-1})=\delta(f)=1$.

-Case 2: $\delta(f)>1$. Then from Theorem 5.1 in [12], $\delta(f)$ is a root of multiplicity 1 and it is the only root λ of $\chi(f)$ with $|\lambda|>1$. The same is true for $\chi(f^{-1})$. Hence using the observation above about roots of $\chi(f)$ and $\chi(f^{-1})$ we conclude that $\chi(f)$ has no other roots of complex norm not equal to 1 other than $\delta(f)$ and $1/\delta(f)$.

Now we complete the proof of a). Assume that $\delta(f)>1$. Then by the remark from the beginning of the proof, $\delta(f)\notin\mathbb{Q}$ (other wise $1>\lambda=1/\delta(f)$ is also a rational root of $\chi(f)(x)$ which is contradict to that remark). Let $p(x)$ be the minimal polynomial of $\delta(f)$. Then it is a root of $\chi(f)$, and all of its coefficients are integers. Now use Case 2 above we now show that $p(x)$ is symmetric. We have

$$\prod_{\alpha:\ p(\alpha)=0}|\alpha|=|p(0)|\geq 1,$$

here one of the α's is $\delta(f)$, another of the α's is $1/\delta(f)$, and the others are algebraic numbers of complex norm 1. This proves a).

b) The proof of b) use the proof of a) and is similar to that of a). □

Theorem 4. *Suppose $N=deg(F)\geq 2$. Assume that the map k_F is birationally conjugate to an automorphism. Then $N=3$, and the map F is that described in Case 2a) of Theorem 3.*

Proof. In Theorems 1 and 3, we constructed spaces Z for which the induced map k_Z is 1-regular, and introduced polynomials $h(x)$ such that $h(x)=(x-1)^2g(x)$ where $g(x)$ factors of the corresponding characteristic polynomials of k_Z which has $\delta(k_F)$ as a root and $g(0)\neq 0$. Hence we can apply Lemma 4 b) to rule out cases for which k_F can not be birationally conjugate to an automorphism.

-Case 1: $N\geq 2$ is even. In this case we have two subcases, which for convenience we list in the same order to that of the statement of Theorem 1:

Subcase 1a) In this case $g(x)=x^{2m+1}(x^2-(N+1)x-1)+x^2+N$. Then $g(x)$ is neither symmetric nor anti-symmetric. Hence by Lemma 4, k_F does not conjugate with an automorphism.

Subcase 1b): In this case $g(x)=x^2-(N+1)x-1$. Although in this case $g(x)$ is anti-symmetric, we see that $g(x)$ is irreducible and has two roots λ and $-1/\lambda$. If k_F was to be conjugate to an automorphism, by Lemma 4, the two roots of $g(x)$ should be λ and $1/\lambda$. Hence in this case k_F does not conjugate to an automorphism.

-Case 2: $N\geq 3$ is odd. We have several subcases, which for convenience we list in the same order that of the statement of Theorem :

Subcase 2.1 a) In this case $g(x) = (1+x^{2m+1})[x^3 - Nx^2 - (N-h+1)x - 1] + (N+1)x^2 + (2N-h+1)x + N - h$. Since $N - h \geq 2$, $g(x)$ is neither symmetrical or anti-symmetrical. Hence as in Case 1a), k_F does not conjugate to an automorphism.

Subcase 2.1 b) In this case $g(x) = x^3 - Nx^2 - (N+1-h)x - 1$. In this case $g(x)$ can not be symmetric. Although $g(x)$ can be anti-symmetric but we always have $g(1) = -N - (N+1-h) < 0$. Hence from Lemma 4, it follows that k_F does not conjugate to an automorphism.

Subcase 2.2 a) It is proved in [7] that in this case k_F does conjugate to an automorphism.

For the other subcases 2.2 b, c, d, it can be easily seen that $g(x)$ is neither symmetric nor anti-symmetric. Hence in these cases, k_F does not conjugate to an automorphism. □

6. Appendix 1: A system of linear equations

In this section we explore the system of linear equations defined in in (5.1). Functions $L_j = L_j(a_0, \ldots, a_n)$ for some first values of j are:

$$\begin{aligned} L_0 &= a_n + [-a_n] = 0, \\ L_1 &= (a_n + a_{n-1}) + [-na_n + a_{n-1}] = -(n-1)a_n + 2a_{n-1}, \\ L_2 &= (a_{n-1} + a_{n-2}) + [-a_n\binom{n}{2} + a_{n-1}\binom{n-1}{1} - a_{n-2}\binom{n-2}{0}] = \frac{n}{2}L_1. \end{aligned}$$

We will explore the properties of systems of linear equations of the form

$$L_j(a_0, a_1, \ldots, a_n) = 0 \tag{6.1}$$

for all $j = 0, 1, 2, \ldots, m$, where $0 \leq m < n$ is a constant integer. It will be convenient to write equations (6.1) as

$$-(a_{n-j} + a_{n-j+1}) = -a_n\binom{n}{j} + a_{n-1}\binom{n-1}{j-1} + \ldots + (-1)^{j+1}a_{n-j}\binom{n-j}{0} \tag{6.2}$$

Changing the order of indexes ($b_j := a_{n-j}$), the equations (6.2) can be written in a more convenient form

$$-(b_j + b_{j-1}) = -b_0\binom{n}{j} + b_1\binom{n-1}{j-1} + \ldots + (-1)^{j+1}b_j\binom{n-j}{0}. \tag{6.3}$$

Lemma 5. *If $0 \leq m < n$, and m is odd, and if $b_0, b_1, \ldots, b_n$ satisfy the equations (6.3) for all $j = 1, 3, 5, \ldots, m$ then $b_0, b_1, \ldots, b_n$ also satisfy (6.3) for all $j = 0, 2, 4, \ldots, m+1$.*

Proof. Fixed $0 \leq m < n$, where m is odd. Let $b_0, b_1, \ldots, b_n$ satisfy the equations (6.3) for all $j = 1, 3, 5, \ldots, m$. To prove Lemma 5 it suffices to prove the following claim:

Claim 1: $b_0, b_1, \ldots, b_n$ also satisfy (6.3) for $j = m+1$.

The proof is divided in several steps.

i) Reduction 1: In equations (6.3) with $j = 1, 3, \ldots, m$, pushing all b_i with i odd to the left hand-sided and pushing all b_i with i even to the right hand-sided we can rewrite them as

$$
\begin{aligned}
&2b_1 = b_0\binom{n-1}{1},\\
&b_1\binom{n-1}{2} + 2b_3 = b_0\binom{n}{3} + b_2\binom{n-3}{1},\\
&b_1\binom{n-1}{4} + b_3\binom{n-3}{2} + 2b_5 = b_0\binom{n}{5} + b_2\binom{n-2}{3} + b_4\binom{n-5}{1},\\
&\vdots\\
&b_1\binom{n-1}{m-1} + b_3\binom{n-3}{m-3} + \ldots + b_{m-2}\binom{n-m+2}{2} + 2b_m\\
&= b_0\binom{n}{m} + b_2\binom{n-2}{m-2} + \ldots + b_{m-3}\binom{n-m+3}{3} + b_{m-1}\binom{n-m}{1}.
\end{aligned}
$$

The equation (6.3) for $j = m+1$ which we want to prove in Claim 1 can be written as

$$
\begin{aligned}
& b_1\binom{n-1}{m} + b_3\binom{n-3}{m-2} + \ldots + b_{m-2}\binom{n-m+2}{3} + b_m\binom{n-m+1}{1}\\
= \;& b_0\binom{n}{m+1} + b_2\binom{n-2}{m-1} + \ldots + b_{m-1}\binom{n-m+1}{2}.
\end{aligned}
$$

ii) Reduction 2: For any value of $b_0, b_2, b_4, \ldots, b_{m-1}$ there exists a unique solution $b_1, b_3, \ldots, b_m$ to the system (6.3) for $j = 1, 3, \ldots, m$. For a proof of this claim we can use the rewritten system in Reduction 1.

iii) Reduction 3: Claim 1 is true in general case if we can show that it is true for the special case $b_0 = 1, b_2 = b_4, \ldots = 0$. For a proof use the special structure of the rewritten system in Reduction 1.

From now on in this proof we will assume that $b_0 = 1, b_2 = b_4 = \ldots = 0$. We rewrite Reduction 1 as

iv) Reduction 4: In equations (6.3) with $j = 1, 3, \ldots, m$, pushing all b_i with i odd to the left hand-sided and pushing all b_i with i even to the right hand-sided we can rewrite them as

$$
\begin{aligned}
2b_1 &= \binom{n-1}{1},\\
b_1\binom{n-1}{2} + 2b_3 &= \binom{n}{3},\\
b_1\binom{n-1}{4} + b_3\binom{n-3}{2} + 2b_5 &= \binom{n}{5},\\
&\vdots\\
b_1\binom{n-1}{m-1} + b_3\binom{n-3}{m-3} + \ldots + b_{m-2}\binom{n-m+2}{2} + 2b_m &= \binom{n}{m}.
\end{aligned}
$$

The equation (6.3) for $j = m+1$ which we want to prove in Claim 1 can be written as

$$
b_1\binom{n-1}{m} + b_3\binom{n-3}{m-2} + \ldots + b_{m-2}\binom{n-m+2}{3} + b_m\binom{n-m+1}{1} = \binom{n}{m+1}.
$$

v) Reduction 5: Define

$$\begin{aligned}\beta_1 &= \frac{b_1}{n},\\ \beta_3 &= \frac{b_3}{n(n-1)(n-2)},\\ \beta_5 &= \frac{b_5}{n(n-1)(n-2)(n-3)(n-4)},\\ &\ldots\end{aligned}$$

then $\beta_1,\beta_3,\beta_5,\ldots$ satisfy the following system of equations

$$\begin{aligned}2\beta_1 &= 1-\frac{1}{n},\\ \frac{\beta_1}{2!}+2\beta_3 &= \frac{1}{3!},\\ \frac{\beta_1}{4!}+\frac{\beta_3}{2!}+2\beta_5 &= \frac{1}{5!},\\ &\ldots,\\ \frac{\beta_1}{(m-1)!}+\frac{\beta_3}{(m-3)!}+\ldots\frac{\beta_{m-2}}{2!}+2\beta_m &= \frac{1}{m!}.\end{aligned}$$

What we want to prove in Claim 1 can be written as

$$\frac{\beta_1}{m!}+\frac{\beta_3}{(m-2)!}+\ldots+\frac{\beta_{m-2}}{3!}+\beta_m(1+\frac{1}{n-m})=\frac{1}{(m+1)!}$$

vi) Reduction 6: A universal system of linear equations

Let $\theta_1,\theta_3,\theta_5,\ldots$ be the unique sequence satisfying the following system of infinitely many linear equations

$$\begin{aligned}2\theta_1 &= 1,\\ \frac{\theta_1}{2!}+2\theta_3 &= 0,\\ \frac{\theta_1}{4!}+\frac{\theta_3}{2!}+2\theta_5 &= 0,\\ &\ldots,\end{aligned}$$

Then, for any sequence $c_1,c_3,c_5,\ldots$, the unique solution to

$$\begin{aligned}2z_1 &= c_1,\\ \frac{z_1}{2!}+2z_3 &= c_3,\\ \frac{z_1}{4!}+\frac{z_3}{2!}+2z_5 &= c_5,\\ &\ldots,\end{aligned}$$

is

$$\begin{aligned}z_1 &= c_1\theta_1,\\ z_3 &= c_3\theta_1+c_1\theta_3,\\ z_5 &= c_5\theta_1+c_3\theta_3+c_5\theta_1,\\ &\ldots\end{aligned}$$

vii) Reduction 7: Let $\alpha_1, \alpha_3, \ldots$ be the unique sequence satisfying the following system

$$\begin{aligned} 2\alpha_1 &= \frac{1}{1!}, \\ \frac{\alpha_1}{2!} + 2\alpha_3 &= \frac{1}{3!}, \\ \frac{\alpha_1}{4!} + \frac{\alpha_3}{2!} + 2\alpha_5 &= \frac{1}{5!}, \\ &\ldots \end{aligned}$$

Then it is easy to see that for β_j in Reduction 4:

$$\beta_j = \alpha_j - \frac{1}{n}\theta_j,$$

for all $j = 1, 3, \ldots, m$, and what we wanted to prove in Claim 1 becomes

$$-\frac{1}{n}\left(\frac{\theta_1}{m!} + \frac{\theta_3}{(m-2)!} + \ldots + \frac{\theta_{m-2}}{3!} + \frac{\theta_m}{1!} - \frac{\theta_m}{m}\right) + \frac{1}{n-m}\left(\alpha_m - \frac{\theta_m}{m}\right) = 0.$$

Hence Claim 1 is proved if we can prove the following Claim

Claim 2: For any $m \in \mathbb{N}$, m odd then the following conclusions are true

$$\frac{\theta_1}{m!} + \frac{\theta_3}{(m-2)!} + \ldots + \frac{\theta_{m-2}}{3!} + \frac{\theta_m}{1!} - \frac{\theta_m}{m} = 0, \tag{6.4}$$

and

$$\alpha_m - \frac{\theta_m}{m} = 0. \tag{6.5}$$

viii) Proof of Claim 2:

Define a formal series

$$\theta(t) = \theta_1 - t^2\theta_3 + t^4\theta_5 - t^6\theta_7 + \ldots$$

From the Reduction 6:

$$1 = \theta(t).(2 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \ldots) = \theta(t).(1 + \cos t).$$

Hence

$$\theta(t) = \frac{1}{1 + \cos t}.$$

Similarly, if we define

$$\alpha(t) = t\alpha_1 - t^3\alpha_3 + t^5\alpha_5 \ldots$$

then from Reduction 7

$$\alpha(t) = \frac{\sin t}{1 + \cos t}.$$

It follows that

$$\frac{d\alpha}{dt} = \theta(t),$$

which proves (6.5).

From Reductions 6 and 7 we have

$$\alpha_m = \frac{\theta_1}{m!} + \frac{\theta_3}{(m-2)!} + \ldots + \frac{\theta_{m-2}}{3!} + \frac{\theta_m}{1!}.$$

This equality and (6.5) imply (6.4). Hence we completed the proof of Lemma 5. $\square$

Lemma 6. *Let $n \geq 3$ be an odd integer. Let $a_0, \ldots, a_n$ be a solution of the system of linear equations*

$$L_j(a_0, a_1, \ldots, a_n) = 0$$

for all $j = 0, 1, 2, \ldots, n-1$. Then

$$\sum_{j=2}^{n} (-1)^j a_j = 0.$$

Proof. To prove the equality we need only to take the difference between the sum of odd-th equations and the sum of even-th equations. □

7. Appendix 2: Proof of Theorem 3

Proof. The proof is divided into some steps.

Step 1: If $h < N-2$, we construct a sequence $Y_1, Y_2, \ldots, Y_{h+1}$ where $Y_{j+1} \to Y_j$ is a blowup of Y_j at a point $\zeta_j \in P_{N-1+j}$, where P_{N-1+j} is the exceptional fiber of the blowup $Y_j \to Y_{j-1}$. Here ζ_j's are constructed inductively in the same way as $\zeta_1, \zeta_2, \zeta_3$ in Section 5. We use the coordinate projection at P_{N-1+j} as follows

$$(u,s) \in Y_j \mapsto [s^N(\zeta_0 + s\zeta_1 + \ldots s^{j-1}\zeta_{j-1} + s^j u) : 1 : s^{N-1}(\zeta_0 + s\zeta_1 + \ldots s^{j-1}\zeta_{j-1} + s^j u)] \in \mathbb{P}^2.$$

The induced map $k_{Y_{h+1}}$ is as follows (see Lemma 2):

$$k_{Y_{h+1}} : C_1, C_2, P_1, \ldots, P_{N-1-(h+1)-1} \mapsto \zeta_{h+1} \in P_{N-1+h+1},$$
$$k_{Y_{h+1}}^{-1} : C_1, P_1, \ldots, P_{N-1-(h+1)-1} \mapsto \xi_{h+1} \in P_{N-1+h},$$

where ζ_{h+1} and ξ_{h+1} are constructed in the same way as $\zeta_1, \zeta_2, \zeta_3$. Moreover $k_{Y_{h+1}} : P_{N-1+(h+1)} \longleftrightarrow P_{N-1-(h+1)}$ is

$$P_{N-1+(h+1)} \ni u \mapsto \frac{(-1)^{N-(h+1)}}{-a_N^2 u + a_N^2 \xi_{h+1}} \in P_{N-1-(h+1)},$$
$$P_{N-1-(h+1)} \ni u \mapsto \frac{(-1)^{N-(h+1)}}{-a_N^2 u} + \zeta_{h+1} \in P_{N-1+(h+1)}.$$

Step 2: The case when $h = N-2$ can be treated as the case when $a_2 = a_3$ in Theorem 2. We construct a sequence $Y_1, Y_2, \ldots, Y_{N-1}$ as in Step 1. Then the induced map $k_{Y_{N-1}}$ is as follows

$$k_{Y_{N-1}} : P_{N-1+N-1} \longleftrightarrow C_1,$$
$$k_{Y_{N-1}} : C_2 \mapsto \zeta_{N-1} \mapsto e_2 = [0:0:1],$$

where $\zeta_{N-1} \in P_{N-1+N-1}$ is constructed as $\zeta_1, \zeta_2, \zeta_3$. Hence we see that the orbit of the exceptional curve C_2 contains the indeterminacy point e_2.

Let $Y_N \to Y_{N-1}$ be the blowup of two points $\zeta_{N-1} \in P_{N-1+N-1}$ and e_2, and call P_{N-1+N} and E_2 the corresponding exceptional fibers of this blowup. We choose the coordinate projection at P_{N-1+N} as

$$(u,s) \in Y_N \mapsto [s^N(\zeta_0 + s\zeta_1 + \ldots s^{N-1}\zeta_{N-1} + s^N u) : 1 : s^{N-1}(\zeta_0 + s\zeta_1 + \ldots s^{N-1}\zeta_{N-1} + s^N u)] \in \mathbb{P}^2,$$

and the coordinate projection at E_2 as

$$(u,s) \in Y_N \mapsto [s : su : 1] \in \mathbb{P}^2.$$

Using computations as in Lemma 2 we can show that in case $h = N-2$ then the induced map $k_{Y_N} : P_{2N-1} \longleftrightarrow E_2$ is

$$k_{Y_N} : \; P_{2N-1} \ni u \quad \mapsto \quad -a_N^2 u + a_N^2 \xi_N - (N+1) \in E_2,$$
$$k_{Y_N} : \; E_2 \ni u \quad \mapsto \quad \frac{-u + a_N^2 \zeta_N + 1}{a_N^2} \in P_{2N-1}.$$

Here ζ_N and ξ_N is constructed in similar manner to that of $\zeta_1, \xi_1, \zeta_2, \xi_2$.

That the point $0 \in E_2$ is the unique indeterminacy point of k_{Y_N} lying on E_2 is not hard to see. It also easy to see that C_2 is an exceptional curve for k_{Y_N}. We have $C_2 \cap E_2 = 1 \in E_2$, which is a regular point of the map k_{Y_N}. Hence

$$k_{Y_N}(C_2) = k_{Y_N}([1]_{E_2}) = \zeta_N.$$

The map $k_{Y_N}^2 : E_2 \to E_2$ is $u \mapsto u + a_N^2(\xi_N - \zeta_N) - (N+2)$, and $k_{Y_N}(\zeta_N) = a_N^2(\xi_N - \zeta_N) - (N+1)$.

When $h = N-2$, Lemma 5 implies $L_j = 0$ for all $j = 1, \dots, N-1$. From the formulas for ξ_N and ζ_N, it follows that

$$a_N^2(\xi_N - \zeta_N) = 2a_0 + \sum_{j=2}^{N} (-1)^j a_j.$$

Lemma 6 implies $a_N^2(\xi_N - \zeta_N) = 2a_0$. Hence the orbit of C_2 is

$$k_{Y_N}^{2l+2} : C_2 \mapsto 2a_0(l+1) - (N+1)(l+1) - l \in E_2.$$

Hence this orbit contains a point of indeterminacy point of k_{Y_2} iff that point is $0 \in E_2$, that is iff there exists an integer $l \geq 0$ for which $2a_0(l+1) - (N+1)(l+1) - l = 0$. The latter condition is exactly the cases 5 and 6 of Theorem 3. If this is the case, then we construct a space Z as the blowup of Y_N at the points $\zeta_N \in P_{N-1+N}, k_{Y_N}(\zeta_N) \in E_2, k_{Y_N}^2(\zeta_N) \in P_{N-1+N}, \dots, k_{Y_N}^{2l+1}(\zeta_N) = 0 \in E_2$ as in the proof of of Theorem 2. Then the induced map k_Z is good, that is it satisfies $(k_Z^*)^n = (k_Z^n)^*$ for all integer $n \geq 0$. Hence the spectral radius of k_Z^* is $\delta(k_F)$. □

References

[1] N. Abarenkova, J-C. Angles dAuriac, S. Boukraa, S. Hassani and J-M. Maillard, *From Yang-Baxter equations to dynamical zeta functions for birational transformations.* Statistical physics on the eve of the 21st century, 436490, Ser. Adv. Statist. Mech., 14, World Sci. Publishing, River Edge, NJ, 1999.

[2] N. Abarenkova, J-C. Angles dAuriac, S. Boukraa, S. Hassani and J-M. Maillard,*Rational dynamical zeta functions for birational transformations,* Physica A 264 (1999) pp. 264–293. chao-dyn/9807014.

[3] J. C. Angles d'Auriac, J. M. Maillard, and C. M. Viallet, *A classification of four-state spin edge Potts models,* J. Phys. A 35 (2002), 9251–9272.

[4] J. C. Angles d'Auriac, J. M. Maillard, and C. M. Viallet, *On the complexity of some birational transformations,* arXiv: math-phys/0503074.

[5] E. Bedford and J. Diller, *Dynamics of a Two Parameter Family of Plane Birational Maps: Maximal entropy,* J. of Geom. Analysis, 16 (2006), no. 3, 409–430.

[6] E. Bedford and K-H Kim,*Degree growth of matrix inversion: birational maps of symmetric, cyclic matrices,* Discrete Con. Dyn. Syst., to appear. arxiv:math.DS/0512507.

[7] Eric Bedford, Kyounghee Kim, Tuyen Trung Truong, Nina Abarenkova, and Jean-Marie Maillard, *Degree complexity of a family of birational maps,* Math Phys Anal Geom 11 (2008), pp. 53–71 .

[8] M. P. Bellon, J-M. Maillard, and C. M. Viallet, *Integrable Coxeter groups,* Phys. Lett. A 159 (1991), pp. 221–232.

[9] M. Bellon and C. M. Viallet, *Algebraic entropy,* Comm. Math. Phys. 204 (1999), 425–437.

[10] S. Boukraa, S. Hassani and J-M. Maillard, *Product of involutions and fixed points,* Alg. Rev. Nucl. Sci., Vol. 2 (1998), 116.

[11] S. Boukraa and J-M. Maillard, *Factorization properties of birational mappings,* Physica A 220 (1995), pp. 403–470.

[12] J. Diller and C. Favre, *Dynamics of bimeromorphic maps of surfaces,* Amer. J. Math., 123 (2001), 1135–1169.

[13] J-E. Fornaess and N. Sibony, *Complex dynamics in higher dimension, II,* Modern Methods in Complex Analysis, Ann. of Math. Studies, vol. 137, Princeton U. Press, 1995, p. 135–182.

Department of mathematics, Indiana University Bloomington, IN 47405
E-mail address: truongt@indiana.edu