Research Statement
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This is the short version. Here I do not include the section "Ongoing and future research", which is available in the long version.

1. Introduction

My research lies in Complex and non-Archimedean Dynamics, Complex and Algebraic Geometry, Pluripotential Theory, and Several Complex Variables. The main objects of my research are complex varieties or projective varieties over a field other than complex numbers (for example the field of p-adic numbers), their geometric properties, and (meromorphic or rational) maps between them. My study in these fields has applications to Statistical Mechanics (see the papers [8, 13, 73]), answering several questions asked by my coauthor Jean-Marie Maillard, a researcher in Mathematical Physics, and his collaborators. One of the highlights of my work is recent results jointly obtained with Fabrizio Catanese and Keiji Oguiso on a challenging question posed in 1975 by Kenji Ueno and on automorphisms of 3-folds [20, 61, 62]. In fact, our results will be presented in the invited talk of my coauthor Keiji Oguiso at the International Congress of Mathematicians 2014.

An essential part of my research is Complex Dynamics, which is a part of the larger theory of Dynamical Systems. It studies dynamical properties (such as topological entropy, invariant measures of maximal entropy, Lyapunov exponents, distribution of periodic points and preimages, Fatou and Julia sets) of holomorphic and, more generally, meromorphic selfmaps. This active field lies in the intersection of many branches of mathematics such as Complex Analysis, Dynamical Systems, Ergodic Theory, Complex Geometry, and Algebraic Geometry. Complex Dynamics has applications to the classification of complex and projective varieties, for example it helps to find varieties having "large" automorphism or bimeromorphic group. In the one dimensional case, any meromorphic selfmap of a smooth projective curve is holomorphic, hence continuous. However, in higher dimensions, a general meromorphic selfmap is not even measurable because of the indeterminate points where it is not defined. This makes it difficult to apply directly the classical ergodic theory. In the course of investigating dynamics of meromorphic maps of complex varieties, I have employed many mathematical tools and ideas from various other branches of mathematics such as positive closed currents, pullback on cohomology groups, Galois theory, intersection theory, Chow groups, and spectral sequences. In turn, my work on dynamics of meromorphic maps has let me to resolve a number of open and interesting problems in other fields. For example, in [20, 61] Catanese, Oguiso and I establish the unirationality and rationality of some varieties which are quotients of projective 3-tori (see Subsections 2.1 and 2.2) motivated by our study of automorphisms of positive entropy of 3-folds. In addition, my joint work with Dinh and Nguyen on dynamical degrees [33] gives constraints on the geometry of varieties having "interesting" selfmaps and on meromorphic maps having invariant fibrations (see Subsection 3.1). Recently, in a joint project with Charles Favre, I apply my results in [71] to investigate dynamics over non-Archimedean fields (for instance, the field $\mathbb{Q}_p$ of p-adic numbers).

I shall now discuss briefly some of my selected mathematical contributions. Further details will be presented in the next sections.

Automorphism of positive entropy + (Uni)rational varieties. A projective variety X is rational if it is birationally equivalent to a projective space $\mathbb{P}^N$. X is unirational if there is a dominant rational map from a projective space to X. The question of whether a variety X is rational or unirational is fundamental in Algebraic Geometry. The automorphism or birational group of a variety (in particular a rational variety) is also another stimulating topic. It is generally very difficult to determine whether a given projective variety is (uni)rational and to construct automorphisms of positive entropy of complex varieties. Here, by Mikhail Gromov and Yosef Yomdin's results (see Section 3), the topological entropy of a surjective holomorphic map $f : X \to X$ is simply

$\log \max_{1\leq p\leq dim(X)} r_p(f)$ where $r_p(f)$ is the spectral radius of the pullback map f^* on the Dolbeault cohomology group $H^{p,p}(X)$.

In his book [75] on classification of algebraic and complex varieties published in 1975, Kenji Ueno asked whether a specific finite quotient Y of a complex 3-torus is unirational or not. In [17], after showing that Y is rationally connected (this means that two general points of Y can be connected by a rational curve belonging to Y), Frederic Campana asked again whether Y is unirational. He suggested that Y is a good test case to check questions on (uni)rationality and rationally connectedness. Recently, together with Catanese and Oguiso [20], I solved this question, confirming that the variety is indeed unirational. The method in [20] is inspired by that in my previous joint work with Keiji Oguiso [61], in which I tackled a different question: is there a rational 3-fold having automorphisms of positive entropy which do not preserve any non-trivial meromorphic fibration? We gave, for the first time, examples answering this question in the affirmative. These examples are still the only known ones up-to-date. The approach in our papers [20, 62] is also promising in constructing examples in higher dimensions. Besides this approach using quotients of complex tori, there is another approach using iterated blowups of projective spaces $\mathbb{P}^N$. While the later approach is very successful in dimension 2 (see Section 2), I showed in [72, 70] that it seems not be promising in dimension at least 3.

We recall that an algebraic number λ is Salem if $\lambda > 1$ is the root of an irreducible monic polynomial $p(t) \in \mathbb{Z}[t]$ whose roots are λ, $1/\lambda$, and complex numbers with absolute value 1. It is well-known that the topological entropy of an automorphism of a complex surface is either 0 or the logarithm of a Salem number (see Subsection 2.3). In joint work with Keiji Oguiso [62], I found a similar relation for automorphisms of complex 3-tori having equivariant holomorphic fibrations. Salem numbers also appear in some other known examples of meromorphic maps in dimension 3 (see Subsection 3.2), and it is quite exciting to see to what extent does this relation between Salem number and dynamics of meromorphic maps (in particular in dimension 3) hold?

Distribution of isolated periodic points of meromorphic maps. For a holomorphic selfmap $f : X \to X$, its periodic points of period n are exactly the points in the intersection of the graph Γ_n of f^n and the diagonal Δ_X. Here we use the convention that f^n is the n-th iterate of f. Because Γ_n and Δ_X have complementary dimensions in $X \times X$, we expect that f has only isolated periodic points. However, since Γ_n and Δ_X may not intersect properly, the set of periodic points may not be isolated. Then it is an interesting and difficult problem to understand the set of isolated periodic points of a holomorphic selfmap and more generally meromorphic selfmap. There is a folklore conjecture saying that if a meromorphic selfmap $f : X \to X$ of a compact Kähler manifold has a dynamical degree larger than other dynamical degrees (for the definition of dynamical degrees, please see Section 3), then f has good ergodic properties. In particular, for such an f, the isolated periodic points should be equidistributed with respect to an equilibrium measure of f and Zariski dense. The latter means that there is no proper subvariety of X containing all isolated periodic points of f. In joint work with Dinh and Nguyen [32], I verify this conjecture for meromorphic maps f for which the topological degree is strictly larger than other dynamical degrees. Our result applies in particular to the case of a polarized endomorphism of positive entropy of a projective variety X, i.e. an endomorphism $f : X \to X$ such that $f^*(L) \simeq L^d$ for an ample line bundle L on X and a number $d > 1$. Here we allow X to be singular, by applying the result to the induced map on any resolution of singularities of X .

In an ongoing project, Tien-Cuong Dinh, Viet-Anh Nguyen and I made some progress extending the results in our paper [32] to other classes of meromorphic maps. For example, we are able to show that if $g : \mathbb{C}^2 \to \mathbb{C}^2$ is a polynomial map with $\lambda_1(g) \neq \lambda_2(g)$ (here λ_1 and λ_2 are the first and second dynamical degrees, for more details please see Section 3), then its number of isolated periodic points of period n is approximately $\max\{\lambda_1(g)^n, \lambda_2(g)^n\}$ as $n \to \infty$, and its periodic points are Zariski dense. Examples of the form $g(x,y) = (P(x), y+1)$, with $\lambda_1(g) = \lambda_2(g) = deg(P)$ but without periodic points in $\mathbb{C}^2$, show that the result is sharp.

Dynamical degrees. One important tool in Complex Dynamics is dynamical degrees. They are bimeromorphic invariants of a meromorphic selfmap $f : X \to X$ of a compact Kähler manifold X. The p-th dynamical degree $\lambda_p(f)$ is the exponential growth rate of the spectral radii of the pullbacks $(f^n)^*$ on the Dolbeault cohomology group $H^{p,p}(X)$ (please see Section 3 for more details). For a surjective holomorphic map f, the dynamical degree $\lambda_p(f)$ is simply the spectral radius of $f^* : H^{p,p}(X) \to H^{p,p}(X)$. Fundamental results of Gromov [44] and Yomdin [77] expressed the topological entropy of a surjective holomorphic map in terms of its dynamical degrees: $h_{top}(f) = \log \max_{0 \leq p \leq dim(X)} \lambda_p(f)$. Since then, dynamical degrees have played a more and more important role in dynamics of meromorphic maps. In many results and conjectures in Complex Dynamics in higher dimensions, dynamical degrees play a central role. Recently Joseph Silverman (see [66]) proposed several deep conjectures concerning dynamical degrees and their arithmetic analogies to study dynamics of rational maps over a number field. This topic is a new and promising one. Many fundamental questions are still unanswered.

To study dynamics of rational maps over a field other than $\mathbb{C}$ (e.g. the field $\mathbb{Q}_p$ of p-adic numbers), it is desirable to have a purely algebraic method to define dynamical degrees. In [71], I used tools such as algebraic cycles and intersection theory to define dynamical degrees over an arbitrary algebraic closed field of characteristic zero. In an ongoing joint project with Charles Favre, I apply this method to study dynamics over a non-Archimedean field.

Dynamical degrees behave well not only under bimeromorphic maps, but also under a general invariant fibration. In joint work with Dinh and Nguyen [33], I show that if $f : X \to X$, $g : Y \to Y$ and $\pi : X \to Y$ are dominant meromorphic maps of compact Kähler manifolds such that $\pi \circ f = g \circ \pi$, then there are constraints between the dynamical degrees of f and g. This result has applications to the classification of complex varieties, giving an easy proof of a folklore conjecture. The conjecture says that if a compact Kähler manifold X has a dominant meromorphic selfmap with one dynamical degree strictly larger than other dynamical degrees then its Kodaira dimension is 0 or $-\infty$, and its Albanese map is surjective. We show moreover that such an X must be special in the sense of Campana [19]. Another application of this result is criteria, stated solely in terms of dynamical degrees, to detect that a meromorphic selfmap does not have any non-trivial invariant meromorphic fibration, see e.g. Corollary 2.3.

The first dynamical degree $\lambda_1(f)$ plays an essential role in dynamics of a meromorphic selfmap f, for example in the construction of Green's function and current. Guedj [47] showed that the Green current has good equidistribution properties if the following question has an affirmative answer: If $(f^n)^* = (f^*)^n$ on $H^{1,1}(X)$ for every $n \geq 1$ and $\lambda_1(f) > \lambda_2(f)$, is $\lambda_1(f)$ a simple eigenvalue of $f^* : H^{1,1}(X) \to H^{1,1}(X)$? In [71], I show that this is indeed the case, under the more general and natural condition $\lambda_1(f)^2 > \lambda_2(f)$.

In the following five sections, I will present more details of my research. The first four sections (consisting of three sections on the three topics described above plus one section on positive closed currents) discuss the results I obtained in accomplished projects. The last section is devoted to my ongoing and future research.

2. Automorphisms of positive entropy+(Uni)rational varieties

One main goal of Complex Dynamics is to construct examples with good dynamical properties, such as the existence of an equilibrium measure and the equidistribution of periodic points with respect to that equilibrium measure (see Section 4 for more details on periodic points). In this section, I concentrate on one of the best classes, that of automorphisms of positive entropy. To avoid trivial examples such as products of smaller dimensional ones, we require that the automorphisms are primitive. Here we use the following definition by Nakayama and Zhang [60].

Definition 2.1. *A dominant rational map $f : X \to X$ is called imprimitive if there are dominant rational maps $\pi : X \to Y$ and $g : Y \to Y$ such that $\pi \circ f = g \circ \pi$ and $0 < dim(Y) < dim(X)$. If f is not imprimitive then we call it primitive.*

The main question is

Question 2.2. *Given X a projective manifold, is there a holomorphic automorphism $f : X \to X$ of positive entropy which is primitive?*

To detect whether an automorphism is primitive, the following consequence of Theorem 3.1 is very helpful. Recall that λ_p is the p-th dynamical degree, please see Section 3 for more details.

Corollary 2.3 (see Oguiso and Truong [62])**.** *Let X be a compact Kähler manifold of dimension $k \geq 3$, and $f : X \to X$ a bimeromorphism. If either $\lambda_1(f) > \lambda_2(f)$ or $\lambda_{k-1}(f) > \lambda_{k-2}(f)$, then f is primitive.*

It is classical that there are no automorphisms of positive entropy in dimension 1. In dimension 2, Question 2.2 is now fairly well-understood since the work of Cantat [18], with many examples on blowups of $\mathbb{P}^2$ and $K3$ surfaces constructed by Bedford and Kim [3, 6, 4] and McMullen [55, 56, 57, 58]. No iterated blowup of $\mathbb{P}^3$ having an automorphism of positive entropy has been found.

My contributions toward Question 2.2 (in dimension ≥ 3) are contained in the papers [20, 62, 61, 72, 70]. I will describe them briefly in the following subsections.

2.1. Examples on a rational 3-fold.

Despite the abundance of automorphisms of positive entropy on rational surfaces which are blowups of $\mathbb{P}^2$, no example has been found on blowups of $\mathbb{P}^3$. Indeed, my work (which will be presented in Subsection 2.4) suggests that no such example exists. We then can ask whether an example exists on a rational 3-fold, which may not be a blowup of $\mathbb{P}^3$. In cooperation with Keiji Oguiso, I provided the first examples which are still the only ones up to date.

Theorem 2.4 (Oguiso and Truong [61])**.** *There is a rational 3-fold X and an automorphism $f : X \to X$ which is primitive and of positive entropy.*

X is the canonical resolution of a quotient of a 3-torus (for canonical resolution see page 199 in [75]), similar to Ueno's example in Subsection 2.2. Here we take the period to be a 3-th root of unity ω, and the quotient is by the diagonal action of $-\omega$ which has order 6. It is birationally equivalent to a hypersurface of $\mathbb{P}^4$ having a conic bundle with a rational section. Note that provided $n \geq 6$ the similar quotient in higher dimension has Kodaira dimension 0, hence it is not even uniruled.

2.2. **Examples on a unirational 3-fold and a question by Kenji Ueno.** We recall that a projective variety X is unirational if there is a dominant rational map $\mathbb{P}^N \to X$ from a projective space to X. In his book published in 1975, Ueno ([75], page 207) constructed the following example. Let $E_{\sqrt{-1}}$ be the elliptic curve of period $\sqrt{-1}$, i.e. $E_{\sqrt{-1}}$ is the unique (upto isomorphism) elliptic curve having an automorphism τ such that $\tau^* : H^{1,0}(E_{\sqrt{-1}}) \to H^{1,0}(E_{\sqrt{-1}})$ is the multiplication by $\sqrt{-1}$. For simplicity, we denote τ by $\sqrt{-1}$. We let $Y = (E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}})/(\sqrt{-1}, \sqrt{-1}, \sqrt{-1})$ be the quotient of the 3-torus $E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}}$ by the diagonal action of $\sqrt{-1}$. He asked the question whether the 3-fold Y is unirational. This question was still open until recently, the most recent progress was that of Campana [17] in 2012, where he showed that Y is rationally connected. He asked again the question whether Y is unirational.

In collaboration with Catanese and Oguiso, I solve this question.

Theorem 2.5 (Catanese, Oguiso and Truong [20])**.** *Y is unirational.*

The proof of this result is inspired by that of Theorem 2.4. Similar to X in Theorem 2.4, Y is birationally equivalent to a hypersurface of $\mathbb{P}^4$ having a conic bundle. However, this conic bundle has no rational section. The canonical resolution of Y is smooth, unirational, and has primitive automorphisms of positive entropy.

2.3. **Automorphisms of complex 3-tori and Salem numbers.** The first dynamical degree of an automorphism of a Kähler surface is either 1 or a Salem number. In joint work with Oguiso, I find that Salem numbers also play an essential role in dynamics of automorphisms of complex 3-tori. We say that an automorphism $f : X \to X$ has a non-trivial equivariant holomorphic fibration if it is imprimitive in the sense of Definition 2.1 where moreover the map π is holomorphic. We recall that the Picard number of X is the rank of the free Abelian group $N^1(X)$ of divisors of X modulo numerical equivalence.

Theorem 2.6 (Oguiso and Truong [62])**.** *1) Let X be a complex 3-torus, and let $f : X \to X$ be an automorphism with $\lambda_1(f) = \lambda_2(f) > 1$.*

a) Then f has a non-trivial equivariant holomorphic fibration iff $\lambda_1(f)$ is a Salem number.

b) If $\lambda_1(f)$ is not a Salem number then the Picard number of X is either 0, 3 or 9. In this case X is projective iff its Picard number is 9.

2) If $r \in \{0, 3, 9\}$, there is a complex 3-torus X of Picard number r having an automorphism $f : X \to X$ such that $\lambda_1(f) = \lambda_2(f) > 1$ is not a Salem number.

In [62] we also prove some other results concerning automorphisms of complex tori in any dimension. The proof of Theorem 2.6 uses my joint result with Oguiso mentioned in Subsection 3.2, an investigation of the Galois group G over $\mathbb{Q}$ of the characteristic polynomial of $f^* : H^{1,0}(X) \to H^{1,0}(X)$, and the action of G on $H^{1,1}(X)$.

2.4. **Automorphisms of blowups of projective manifolds.** The following question was asked by Eric Bedford in 2011: Is there an iterated sequence of blowups of $\mathbb{P}^3$ along smooth centers, such that the resulting manifold X has an automorphism f with positive entropy? While there are several constructions of pseudo-automorphisms on blowups of $\mathbb{P}^3$ with $\lambda_1(f) > 1$, no automorphism was found. In fact, in [72], I constructed many examples of iterated blowups of $\mathbb{P}^3$ having no automorphism of positive entropy. Similar results also apply for blowups of $\mathbb{P}^2 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

For example, if S is an iterated blowup of $\mathbb{P}^2$, then for any automorphism f of $S\times\mathbb{P}^1$ we have $\lambda_1(f)=\lambda_2(f)$. This suggests that probably all automorphisms of $S\times\mathbb{P}^1$ are of product type.

Later on, Bayraktar and Cantat [2] partially extended results in [72] to blowups X of projective manifolds of Picard number 1 of higher dimensions. Their condition is that the centers of blowups must have dimension $<(dim(X)-2)/2$ (hence e.g. when the dimension is 3, their results can be applied for point blowups), and their conclusion is that $Aut(X)$ has only a finite number of connected components. Note that the latter implies that all automorphisms of X have zero topological entropy. In my recent work [70], I extended these results to blowups of other manifolds, including Fano and projective hyper-Kähler manifolds. The conditions used in this paper (as well as in [72]) are based on properties of nef cohomology classes, and are satisfied under conditions of the results in [2]. Recall that a cohomology class $\zeta\in H^{1,1}(X)$ is nef if it can be represented as a limit of a sequence of Kähler forms. I prove, for example, the following.

Theorem 2.7 (Truong [70]). *Let X be a compact Kähler manifold of dimension $k\geq 3$. Let K_X be the canonical divisor of X. Assume that there are integers $-1\leq r\leq k-1$ and $0\leq q\leq k-r-1$ such that for any non-zero nef cohomology class ζ on X, then $\zeta^{k-r-1-q}.K_X^q$ is non-zero. Then $Aut(X)$ has only finitely many connected components.*

In [72, 70], I provide many examples of compact Kähler manifolds which satisfy the condition of Theorem 2.7 or similar conditions.

3. Dynamical degrees

A meromorphic map $f:X\rightarrow Y$ between two complex manifolds is a holomorphic map $f|_U:U\rightarrow Y$ from a Zariski open dense set U of X into Y so that the closure of the graph of $f|_U$ is an analytic subvariety of $X\times Y$. We say that f is dominant if $f|_U(U)$ is dense in Y. An example of rational maps is a rational function f on $\mathbb{C}$ of the form $f(z)=P(z)/Q(z)$ where P and Q are polynomials in the variable $z\in\mathbb{C}$ which are relatively prime. While f is not a continuous selfmap on $\mathbb{C}$, it becomes a holomorphic selfmap when extended to the complex projective line $\mathbb{P}^1$. f has two dynamical degrees: $\lambda_0(f)=1$ and $\lambda_1(f)=\max\{deg(P),deg(Q)\}$ the topological degree of f. The importance of these dynamical degrees was shown already since the works of Gaston Julia, Pierre Fatou and Lucjan Böttcher more than 100 years ago (see Milnor's book [59]). For example, a classical result says that if $\lambda_1(f)>1$ then the periodic points of f are equidistributed with respect to its equilibrium measure (see Section 4 for more details).

Since the fundamental results of Mikhail Gromov and Yosef Yomdin, dynamical degrees have proved important in dynamics of holomorphic and meromorphic selfmaps in higher dimensions as well. In many results and conjectures in complex dynamics in higher dimensions, dynamical degrees play a central role. Let X be a compact Kähler manifold of dimension k, and $f:X\rightarrow X$ a dominant meromorphic map. The pullback maps $f^*:H^{p,p}(X)\rightarrow H^{p,p}(X)$ are well-defined. The idea is that if θ is a smooth form then we can define $f^*(\theta)$ as a current, whose restriction to the open set where f is holomorphic is the usual pullback. Note that in general we do not have the compatibility of pullback maps: $(f^n)^*$ may be different from $(f^*)^n$ on $H^{p,p}(X)$. Russakovskii and Shiffman [65] (the case $X=\mathbb{P}^k$) and Dinh and Sibony [34, 35] (the case of compact Kähler manifolds) defined the p-th dynamical degree $\lambda_p(f)$ as follows

$$\lambda_p(f)=\lim_{n\rightarrow\infty}r_p(f^n)^{1/n}, \tag{1}$$

where $r_p(f^n)$ is the spectral radius of $(f^n)^*: H^{p,p}(X) \to H^{p,p}(X)$. The existence of the limit in (1) is a non-trivial fact. Here are some simple properties of dynamical degrees: $\lambda_0(f) = 1$, $\lambda_k(f) =$ the topological degree of f, $\lambda_{p-1}(f)\lambda_{p+1}(f) \leq \lambda_p(f)^2$ (log-concavity). Dinh and Sibony [34, 35] proved that dynamical degrees are bimeromorphic invariants.

For a surjective holomorphic map f, $\lambda_p(f)$ is simply $r_p(f)$. In this case, we have $h_{top}(f) = \max_{0\leq p\leq k} \log \lambda_p(f)$ where $h_{top}(f)$ is the topological entropy of f. Gromov [44] proved the inequality $h_{top}(f) \leq \max_{0\leq p\leq k} \log \lambda_p(f)$, and Yomdin [77] proved the reverse inequality. The topological entropy $h_{top}(f)$ of a meromorphic map can be defined similarly but more complicated than that of a holomorphic map, see e.g. Guedj [46]. Gromov's inequality still holds for a meromorphic map (see [34, 35]), but Yomdin's inequality does not hold in general (see [46]).

My contributions on this topic are contained in the papers [33, 71]. I briefly describe them in the following subsections.

3.1. **Dynamical degrees of maps preserving a meromorphic fibration.** In collaboration with Tien-Cuong Dinh and Viet-Anh Nguyen, I proved a relation between dynamical degrees of meromorphic maps which are semi-conjugate. Let $f : X \to X$ and $g : Y \to Y$ be dominant meromorphic maps, where X and Y are compact Kähler manifolds of dimensions k and l, where $k \geq l$. Assume that there is a dominant meromorphic map $\pi : X \to Y$ such that $\pi \circ f = g \circ \pi$. Let ω_X be a Kähler form on X and ω_Y a Kähler form on Y. Relative dynamical degrees $\lambda_p(f|\pi)$ (here $0 \leq p \leq k - l$), which are bimeromorphic invariants, can be defined in the simplest case π is a surjective holomorphic map as follows (see Dinh and Nguyen [31]):

$$\lambda_p(f|\pi) = \lim_{n\to\infty} (\int_X (f^n)^*(\omega_X^p) \wedge \pi^*(\omega_Y^l) \wedge \omega_X^{k-p-l})^{1/n}.$$

We proved the following result.

Theorem 3.1 (Dinh, Nguyen and Truong [33]). *Let X and Y be compact Kähler manifolds of dimensions $k \geq l$. Let $f : X \to X$, $\pi : X \to Y$, and $g : Y \to Y$ be dominant meromorphic maps so that $\pi \circ f = g \circ \pi$. Then for all $0 \leq p \leq k$:*

$$\lambda_p(f) = \max_{\max\{0,p-k+l\}\leq j\leq \min\{p,l\}} \lambda_j(g)\lambda_{p-j}(f|\pi). \quad (2)$$

Dinh and Nguyen [31] proved Theorem 3.1 in the projective case, using that any dominant rational map $\pi : X \to Y$ is (up to a dominant meromorphic map of finite degree) the canonical projection $Y \times \mathbb{P}^{k-l} \to Y$. In the Kähler case this is no longer available, and our main idea is to show that for any dominant meromorphic map $\pi : X \to Y$, the cohomology class of the diagonal Δ_X can be bound by $\omega \wedge \Theta$, where ω is the pullback by the map $\pi \times \pi$ of an (l, l) cohomology class on $Y \times Y$. In the course of the proof, we established a relative semi-regularization for positive closed currents of any bidegree which extends the results of Dinh and Sibony in [35, 34].

An easy consequence of Theorem 3.1 is the confirmation of the following folklore conjecture: If X is a compact Kähler manifold having a dominant meromorphic map $f : X \to X$ with a dynamical degree strictly larger than the other dynamical degrees, then the Kodaira dimension of X is 0 or $-\infty$, and the Albanese map of X is surjective.

3.2. **The simplicity of the first dynamical degree, and a lower bound for the second dynamical degree.** A dominant meromorphic map $f : X \to X$ is called 1-stable if for any n we have $(f^n)^* = (f^*)^n$ as linear maps on $H^{1,1}(X) \to H^{1,1}(X)$. Motivated by applications to proving good properties of the Green currents of a meromorphic map, Guedj [47] asked whether the following is true: If $f : X \to X$ is 1-stable and $\lambda_1(f) > \lambda_2(f)$ then $r_1(f^n) = \lambda_1(f)^n + o(\lambda_1(f)^n)$ as $n \to \infty$ (recall that $r_1(f^n)$ is the spectral radius of $(f^n)^* : H^{1,1}(X) \to H^{1,1}(X)$)? I showed that this is indeed the case, under a weaker and more natural condition that $\lambda_1(f)^2 > \lambda_2(f)$. Recall that by log-concavity we always have $\lambda_1(f)^2 \geq \lambda_2(f)$.

Theorem 3.2 (Truong [71])**.** *Let X be a compact Kähler manifold, and $f : X \to X$ a dominant meromorphic map. Assume that f is 1-stable and $\lambda_1(f)^2 > \lambda_2(f)$. Then $\lambda_1(f)$ is a simple eigenvalue of $f^* : H^{1,1}(X) \to H^{1,1}(X)$. Further, $\lambda_1(f)$ is the only eigenvalue of modulus greater than $\sqrt{\lambda_2(f)}$.*

A variant of this result, where instead of f being 1-stable we ask that f^* preserves the cone of cohomology classes in $H^{2,2}(X)$ which can be represented by positive closed currents, and in the places of $\lambda_1(f)$ and $\lambda_2(f)$ we use $r_1(f)$ and $r_2(f)$, also holds. Only the special cases $dim(X) = 2$ (proved by Diller and Favre [27]) and f is holomorphic (this case is quite easy) were known before.

The main idea of the proof is to show that if $\theta \in H^{1,1}(X, \mathbb{C})$ then $f^*(\theta).f^*(\overline{\theta}) - f^*(\theta.\overline{\theta})$ can be represented by a positive closed current, and then use the Hodge index theorem. Here . is the cup product. Note that for a general meromorphic map f the term $f^*(\theta).f^*(\overline{\theta}) - f^*(\theta.\overline{\theta})$ does not vanish.

One consequence of Theorem 3.2 is the following. We recall that a bimeromorphic map $f : X \to X$ is pseudo-automorphic if both f and its inverse have no exceptional hypersurfaces (an exceptional hypersurface of a map is a hypersurface whose image has codimension at least 2). Together with Keiji Oguiso, I proved the following.

Theorem 3.3 (Oguiso and Truong [62])**.** *Let $f : X \to X$ be a pseudo-automorphism of a compact Kähler manifold. Assume that $\lambda_1(f)$ is a Salem number.*

1) If $dim(X) = 4$ then either $\lambda_1(f) = \lambda_3(f)$ or $\lambda_1(f)^2 = \lambda_2(f)$.

2) If $dim(X) = 3$ then $\lambda_1(f) = \lambda_2(f)$.

It is worth noting that for all known examples of pseudo-automorphisms on blowups of $\mathbb{P}^3$ or $\mathbb{P}^2 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see e.g. [5, 7], [63] and [14]) the first dynamical degrees are Salem numbers.

Another consequence of Theorem 3.2 is to estimate the second dynamical degree of a dominant meromorphic map. While the first dynamical degree may be computed exactly in many examples by making a map 1-stable, there is currently no method to compute the second or higher dynamical degrees. Only in some very special cases e.g. monomial maps or maps all fibers are finite (see [1], [41], [52]) that higher dynamical degrees can be computed effectively. Thus a good estimate of higher dynamical degrees is desirable. Assume that $f : X \to X$ is 1-stable, and let δ be any root of the characteristic polynomial $p(t)$ of $f^* : H^{1,1}(X) \to H^{1,1}(X)$ such that $|\delta| < \lambda_1(f)$. Applying Theorem 3.2, we obtain the following bound on the second dynamical degree: $|\delta|^2 \leq \lambda_2(f) \leq \lambda_1(f)^2$. For example, the growth of $\lambda_2(K_q)$, here K_q is the birational selfmap on the space of $q \times q$ matrices considered in my paper jointly written with Eric Bedford [13], is of order at least q^2 and at most q^4.

3.3. **Dynamical degrees of rational maps defined over an algebraic closed field of characteristic zero.** Inspired by a question of Mattias Jonsson on whether Theorem 3.2 can be extended to other fields, I defined dynamical degrees for rational maps over an arbitrary algebraic closed field K of characteristic zero (for example $K = \mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$). When K is different from $\mathbb{C}$, many tools in complex dynamics are no longer available. In general we do not have smooth forms, Dolbeault cohomology groups, and regularization of positive closed currents. Instead, I use the groups $N^p(X)$ of algebraic p-cycles modulo numerical equivalence (these groups are Abelian of finite rank, and are preserved under pullback by rational maps) and Chow's moving lemma to define dynamical degrees. Here is a summary of the results I obtained. We recall here that if $X \subset \mathbb{P}^N_K$ is a smooth projective variety over K, then we can define for any subvariety $W \subset X$ of pure dimension its degree $deg(W)$ as the degree of W viewed as a subvariety of $\mathbb{P}^N_K$.

Theorem 3.4 (Truong [71])**.** *Let K be an algebraic closed field of characteristic zero, X a smooth projective variety, $f : X \to X$ a dominant meromorphic map. Let H be an ample divisor on X. Then*

1. *For any $0 \leq p \leq dim(X)$, the following limit exists*
$$\lambda_p(f) = \lim_{n\to\infty} (deg((f^n)^*(H^p)))^{1/n}.$$
2. *The dynamical degrees $\lambda_p(f)$ are birational invariants.*

Another algebraic method to define dynamical degrees was given simultaneously by Favre [39]. Having Theorem 3.4 at hand, we can extend Theorem 3.2 and its variant and consequences verbatim to all rational maps over an arbitrary algebraic closed field of characteristic zero. Recently, in collaboration with Charles Favre, I apply Theorem 3.4 to study dynamics on Berkovich spaces (see the last section for more details).

4. EQUIDISTRIBUTION AND DENSITY OF ISOLATED PERIODIC POINTS

Periodic points are a topic of extensive study in many fields of mathematics. One of the most famous results on this topic is the Lefschetz fixed point theorem.

In complex dynamics, periodic points of a "good" meromorphic map are expected to be equidistributed with respect to its equilibrium measure. An equidistribution result for a general meromorphic map may be formulated as follows. Let $f : X \to X$ be a dominant meromorphic map of a compact Kähler manifold. We call x an isolated periodic point of order n of f, if f^n is holomorphic near x and x is an isolated fixed point of the restriction of f^n to a small neighborhood of x. Denote by $\mathcal{P}_n$ the set of isolated periodic points of order n of f counted with multiplicity, and for a point $x \in X$ denote by δ_x the Dirac measure at x. Given μ a probability measure on X, we say that isolated periodic points of f are equidistributed with respect to μ if
$$\lim_{n\to\infty} \frac{1}{|\mathcal{P}_n|} \sum_{x\in\mathcal{P}_n} \delta_x = \mu.$$
Here $|\mathcal{P}_n|$ is the cardinality of $\mathcal{P}_n$.

The equidistribution of periodic points was first proved for rational maps on $\mathbb{P}^1$ with topological degree $d > 1$ by Brolin [16], Freie, Lopes and Mane [43], Lyubich [54] and Tortrat [68]. For Hénon maps in dimension 2, it was proved by Bedford, Lyubich and Smillie [9]. For holomorphic maps on $\mathbb{P}^k$, it was proved by Briend and Duval [15]. Recently, equidistribution results were proved by

Diller, Dulardin and Guedj for some classes of meromorphic maps of surfaces [30] and by Dinh and Sibony for Hénon-type maps in higher dimensions [37].

In joint work with Tien-Cuong Dinh and Viet-Anh Nguyen, I prove the equidistribution for meromorphic maps whose topological degree is larger than other dynamical degrees. (Previously, Guedj [45] stated a weaker result for projective manifolds, however his proof is incomplete.) This result is an extension of that of Briend and Duval [15].

Theorem 4.1 (Dinh-Nguyen-Truong [32])**.** *Let X be a compact Kähler manifold of dimension k, and $f : X \to X$ a dominant meromorphic map with $\lambda_k(f) > \max_{p\neq k} \lambda_p(f)$. Let μ be the equilibrium measure of f, which is defined as*

$$\mu = \lim_{n\to\infty} \frac{1}{\lambda_k(f)^n}(f^n)^*(\Theta),$$

where Θ is any smooth probability measure. Then isolated periodic points of f are equidistributed with respect to μ. Consequently, isolated periodic points of f are Zariski dense in X.

Note that in [32] we also proved the equidistribution and Zariski density for preimages of a generic point. There are two main difficulties in proving Theorem 4.1. The first is that even for holomorphic maps, not all periodic points are isolated. Hence a simple application of Lefschetz fixed point theorem may not work, even for simple manifolds such as iterated of point blowups of $\mathbb{P}^3$. The second is that, the construction of good inverses is more difficult for meromorphic maps than for holomorphic maps. To overcome the first obstacle, we apply the theory of tangent currents developed recently by Dinh and Sibony [37]. To overcome the second obstacle we make use of a special positive closed $(1,1)$ current related to the indeterminate and exceptional sets of the map under consideration.

5. Positive closed currents

Positive closed currents are an important tool in complex analysis and complex geometry. Examples of positive closed currents include positive closed smooth forms and currents of integration over complex varieties. For more details on positive closed currents, please see the book Demailly [25]. Since the pioneer works by Bedford, Lyubich and Smillie (see [11, 12] and [9, 10]) and Fornaess and Sibony [42], and with recent developments by Diller, Dujardin, and Guedj [28, 29, 30] and Dinh and Sibony [37, 36], positive closed currents have become an invaluable tool to study ergodic properties of meromorphic maps. For example, they can be used to construct an equilibrium measure or prove the equidistribution of isolated periodic points and preimages. In this approach, we first construct two invariant positive closed currents T^+ and T^- of complementary bidegrees, then show that they have good enough properties so that we can wedge them to get a measure $\mu = T^+ \wedge T^-$ which is an invariant measure for the map f under consideration. Here, it is important to be able to pullback by a meromorphic map and to wedge positive closed currents. In [74], I defined a reasonable pullback of positive closed currents by meromorphic maps. This definition is compatible with the definitions given previously by other authors.

My contributions on this topic are contained in [23, 74, 69], which I describe briefly below.

5.1. **Upper level sets for Lelong numbers of positive closed currents.** Let X be a complex manifold, and T a positive closed (p,p) current on X. An indication of the complexity of T at a point $x \in X$ is the Lelong number $\nu(T,x)$, which in case $T = [V]$ the current of integration over a complex variety V is not other than the multiplicity of V at x. A fundamental result of Siu [67] says that for any $c > 0$ the set $E_c(T) = \{x \in X : \nu(T,x) \geq c\}$ is an analytic set of codimension at least p. In case X is compact Kähler with a Kähler form ω_X, then $E_c(T)$ has a finite number of components, but it is not true that we can bound the number of the components or the total volume of the components in terms of the mass $||T|| =< T, \omega_X^{dim(X)-p} >$ and the number c only. In joint work with Dan Coman, I show however that a slight modification of this statement is true.

Theorem 5.1 (Coman and Truong [23]). *Let X be a compact Kähler manifold with a Kähler form ω_X, and $c > 0$. For any positive closed (p,p) current T on X, there is an analytic set $V \subset X$ containing $E_c(T)$ such that the codimension of V is at least p, and the volume of V and the number of irreducible components of V are $\leq K$ for some constant K depending only on c and the mass $||T||$.*

This result is proved by using Demailly's regularization theorem [24] for positive closed $(1,1)$ currents and a result of Vigny [76] on the existence of a positive closed $(1,1)$ current of the same Lelong numbers as a given positive closed current of any bidegree. In case $X = \mathbb{P}^k$ or $\mathbb{P}^m \times \mathbb{P}^n$, we obtain better results which are optimal, by using in addition explicit positive closed $(1,1)$ currents coming from homogeneous polynomials and properties of rational normal curves. For example, we prove the following.

Theorem 5.2 (Coman and Truong [23]). *Let T be a positive closed (p,p) current on $\mathbb{P}^n$ (here $0 < p < n$), of the same cohomology class as that of a linear subspace of codimension p of $\mathbb{P}^n$. Then the set $\{x \in \mathbb{P}^n : \nu(T,x) > (p+1)/(p+2)\}$ is contained in a linear subspace of codimension p of $\mathbb{P}^n$.*

Some special cases of Theorem 5.2 were proved by Coman ($p = n-1$) [21], and Coman and Guedj ($p = 1$) [22].

5.2. **Dynamics of pseudo-automorphisms in dimension 3.** Given X a compact Kähler manifold, a bimeromorphic map $f : X \to X$ is pseudo-automorphic if both f and f^{-1} have no exceptional hypersurfaces. Here a hypersurface is exceptional if its image has codimension at least 2. The study of pseudo-automorphisms in dimension 3 is interesting because of several reasons. First, pseudo-automorphisms can be regarded as the second best class of meromorphic selfmaps, after that of automorphisms. While a manifold may not have any interesting automorphism, it may have a lot of interesting pseudo-automorphisms. For example, while there seems to be no automorphisms of positive entropy on any iterated blowup of $\mathbb{P}^3$ (see Subsection 2.4), there are many pseudo-automorphisms with $\lambda_1 > 1$ (see e.g. [5, 7], [63] and [14]). Second, while dynamics of bimeromorphic maps of surfaces and of automorphisms are quite well-understood (see [27] and [36]), not much is known for bimeromorphic maps in dimension ≥ 3. In this regard, pseudo-automorphisms in dimension 3 are a good test case.

Using the pullback of currents by meromorphic maps developed in my dissertation (see [74]) and my results on the simplicity of the first dynamical degree (see Subsection 3.2), in the preprint [69] I made first steps toward understanding the dynamics of pseudo-automorphisms in dimension 3. Among other things, I show that if $f : X \to X$ is a pseudo-automorphism in dimension 3, and if T is a positive closed current on X, then $f^*(T)$ is well-defined and is a difference of two positive closed currents. Moreover, the following equality holds:

$$(f^n)^*(T) = (f^*)^n(T),$$

for any integer n and any positive closed current T. Using this equality, I show that if moreover $\lambda_1(f) > 1$, then there is a non-zero positive closed $(2,2)$ T on X such that either $f_*(T) = \lambda_1(f)T$ or

$f^*(T) = \lambda_2(f)T$. Previously, similar results (using super-potentials developed in [38]) were known only on $X = \mathbb{P}^k$, see Dinh and Sibony [38] and Thelin and Vigny [26].

As a related remark, note that recently Roeder [64] considered the compatibility of pulling back on cohomology groups for a composition map, using intersection theory. Alternative proofs for several results in his paper can be given using the methods in the paper [74].

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