Research Summary
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My research lies in Complex and non-Archimedean Dynamics, Complex and Algebraic Geometry, Pluripotential Theory, and Several Complex Variables. The main objects of my research are complex varieties or projective varieties over a field other than complex numbers (for example the field of p-adic numbers), their geometric properties, and (meromorphic or rational) maps between them. My study in these fields has applications to Statistical Mechanics (see the papers [1, 13]), answering several questions asked by my coauthor Jean-Marie Maillard, a researcher in Mathematical Physics, and his collaborators. One of the highlights of my work is recent results jointly obtained with Fabrizio Catanese and Keiji Oguiso on a challenging question posed in 1975 by Kenji Ueno and on automorphisms of 3-folds [2, 7, 8]. In fact, our results will be presented in the invited talk of my coauthor Keiji Oguiso at the International Congress of Mathematicians 2014.

An essential part of my research is Complex Dynamics, which is a part of the larger theory of Dynamical Systems. It studies dynamical properties (such as topological entropy, invariant measures of maximal entropy, Lyapunov exponents, distribution of periodic points and preimages, Fatou and Julia sets) of holomorphic and, more generally, meromorphic selfmaps. This active field lies in the intersection of many branches of mathematics such as Complex Analysis, Dynamical Systems, Ergodic Theory, Complex Geometry, and Algebraic Geometry. Complex Dynamics has applications to the classification of complex and projective varieties, for example it helps to find varieties having "large" automorphism or bimeromorphic group. In the one dimensional case, any meromorphic selfmap of a smooth projective curve is holomorphic, hence continuous. However, in higher dimensions, a general meromorphic selfmap is not even measurable because of the indeterminate points where it is not defined. This makes it difficult to apply directly the classical ergodic theory.

Recent developments in Complex Dynamics, among them are my contributions described below, provide us with a deeper knowledge of the dynamics of meromorphic maps. Many interesting conjectures and problems were solved, and we have a better understanding on the open problems which are fundamental in the field. There has emerged the important role of dynamical degrees together with the use of tools and ideas from fields such as Complex Geometry, Algebraic Geometry and Berkovich Analytic Geometry. Several important and interesting questions in Algebraic Geometry and Arithmetic Dynamics, concerning for example automorphism groups, periodic points and meromorphic fibrations, were solved using tools and ideas in Complex Dynamics. Thus we expect to see more and more fruitful outcomes resulting from the interaction between Complex Dynamics and other fields, in particular non-Archimedean Dynamics, Arithmetic Dynamics and Algebraic Geometry.

1. Selected mathematical contributions

1.1. **Automorphisms of positive entropy+(Uni)rational varieties.** One main problem of Complex Dynamics is to construct automorphisms of positive entropy. While in dimension 2 this problem is now fairly well-understood since the work of S. Cantat, the situation in dimension at least 3 is still mysterious.

Together with Keiji Oguiso, I showed in [7] that a specific variety, which is a finite quotient of a complex 3-torus, is rational. This gives, for the first time, a rational 3-fold having automorphisms of positive entropy which do not preserve any non-trivial meromorphic fibration.

Together with Fabrizio Catanese and Keiji Oguiso, in [2] I answered a question posed in 1975 by Keiji Ueno asking whether a specific finite quotient of a complex 3-torus, similar to that considered in the paper [7], is unirational. The proof is inspired by that in [7].

In another joint work with Keiji Oguiso [8], I showed that an automorphism $f : X \to X$ of a complex 3-torus with $r_1(f) = r_2(f) > 1$ has a non-trivial equivariant holomorphic fibration iff

$r_1(f)$ is a Salem number. Here $r_j(f)$ is the spectral radius of the pullback $f^* : H^{j,j}(X) \to H^{j,j}(X)$. Moreover, we showed that if f has $r_1(f) = r_2(f) > 1$ but no non-trivial equivariant holomorphic fibration then the Picard number of X must be 0, 3 or 9, and all these cases can be realized.

The following question was asked by E. Bedford in 2011: Is there an iterated sequence of blowups of $\mathbb{P}^3$ along smooth centers, such that the resulting manifold X has an automorphism f with positive entropy? In [12], I constructed many examples of iterated blowups of $\mathbb{P}^3$ having no automorphism of positive entropy. Similar results also apply for blowups of $\mathbb{P}^2\times\mathbb{P}^1$ or $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$. In my recent work [10], I extended these results to blowups of other manifolds, including Fano and projective hyper-Kähler manifolds. The conditions used in this paper (as well as in [12]) are based on properties of nef cohomology classes.

1.2. **Dynamical degrees.** One important tool in Complex Dynamics is dynamical degrees. They are bimeromorphic invariants of a meromorphic selfmap $f : X \to X$ of a compact Kähler manifold X. The p-th dynamical degree $\lambda_p(f)$ is the exponential growth rate of the spectral radii of the pullbacks $(f^n)^*$ on the Dolbeault cohomology group $H^{p,p}(X)$, this definition is given by A. Russakovskii and B. Shiffman (for projective spaces) and T.-C. Dinh and N. Sibony (for compact Kähler manifolds). For a surjective holomorphic map f, the dynamical degree $\lambda_p(f)$ is simply the spectral radius of $f^* : H^{p,p}(X) \to H^{p,p}(X)$. In this case M. Gromov and Y. Yomdin proved that $h_{top}(f) = \log\max_{0\leq p\leq dim(X)}\lambda_p(f)$.

To study dynamics of rational maps over a field other than $\mathbb{C}$ (e.g. the field $\mathbb{Q}_p$ of p-adic numbers), it is desirable to have a purely algebraic method to define dynamical degrees. In [11], I used tools such as algebraic cycles and intersection theory to define dynamical degrees over an arbitrary algebraic closed field of characteristic zero.

Dynamical degrees behave well not only under bimeromorphic maps, but also under a general invariant fibration. In joint work with Tien-Cuong Dinh and Viet-Anh Nguyen [5], I show that if $f : X \to X$, $g : Y \to Y$ and $\pi : X \to Y$ are dominant meromorphic maps of compact Kähler manifolds such that $\pi\circ f = g\circ\pi$, then there are constraints between the dynamical degrees of f and g. Using this, we solved a conjecture saying that if a compact Kähler manifold X has a dominant meromorphic selfmap with one dynamical degree strictly larger than other dynamical degrees then its Kodaira dimension is 0 or $-\infty$, and its Albanese map is surjective. Another application of this result is criteria, stated solely in terms of dynamical degrees, to detect that a meromorphic selfmap does not have any non-trivial invariant meromorphic fibration.

V. Guedj showed that the Green current of a meromorphic map f has good equidistribution properties if the following question has an affirmative answer: If $(f^n)^* = (f^*)^n$ on $H^{1,1}(X)$ for every $n \geq 1$ and $\lambda_1(f) > \lambda_2(f)$, is $\lambda_1(f)$ a simple eigenvalue of $f^* : H^{1,1}(X) \to H^{1,1}(X)$? In [11], I show that this is indeed the case, under the more general and natural condition $\lambda_1(f)^2 > \lambda_2(f)$.

In joint work with Keiji Oguiso [8], I applied results in [11] to show that if $f : X \to X$ is a pseudo-automorphism of a compact Kähler manifold X of dimension $k \leq 4$ such that $\lambda_1(f)$ is a Salem number, then either $\lambda_1(f) = \lambda_{k-1}(f)$ or $\lambda_1(f)^2 = \lambda_{k-2}(f)^2$.

1.3. **Distribution of isolated periodic points of meromorphic maps.** Periodic points are a topic of extensive study in many fields of mathematics. One of the most famous results on this topic is the Lefschetz fixed point theorem.

In complex dynamics, periodic points of a "good" meromorphic map are expected to be equidistributed with respect to its equilibrium measure. In joint work with Dinh and Nguyen [4], I verify this conjecture for meromorphic maps f for which the topological degree is strictly larger than other dynamical degrees. Our result applies in particular to the case of a polarized endomorphism of positive entropy of a projective variety X. Previously, Guedj [6] stated a weaker result for projective manifolds, however his proof is incomplete.

1.4. **Positive closed currents.** Since the pioneer works by E. Bedford-M. Lyubich-J. Smillie and J. E. Fornaess-N. Sibony, and with recent developments by J. Diller-R. Dujardin-V. Guedj and T.C. Dinh-N. Sibony, positive closed currents have become an invaluable tool to study ergodic properties of meromorphic maps.

In joint work with Dan Coman [3], I showed the following result for the upper level set for Lelong number $E_c(T) = \{x \in X : \nu(T, x) > c\}$ of a positive closed (p, p) current T on a compact Kähler manifold X. $E_c(T)$ is contained in a subvariety $V \subset X$ of codimension p such that the volume of V and the number of irreducible components of V are bounded by a constant depending only on c and the mass of T. When X is $\mathbb{P}^N$ or $\mathbb{P}^N \times \mathbb{P}^M$ we obtained several optimal results.

In my dissertation [14], I defined a reasonable pullback of positive closed currents by meromorphic maps. This definition is compatible with the definitions given previously by other authors.

Using my results in [14]) and [11], in the preprint [9] I made first steps toward understanding the dynamics of pseudo-automorphisms in dimension 3. Among other things, I showed that a positive closed $(2, 2)$ current can be pulled back by a pseudo-automorphism and this pullback is compatible with iteration. Using this, I constructed invariant positive closed $(2, 2)$ currents for such maps.

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