EXPLICIT EXAMPLES OF RATIONAL AND CALABI-YAU THREEFOLDS WITH PRIMITIVE AUTOMORPHISMS OF POSITIVE ENTROPY

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Abstract. We present the first explicit examples of a rational threefold and a Calabi-Yau threefold, admitting biregular automorphisms of positive entropy not preserving any dominant rational maps to lower positive dimensional varieties. The most crucial part is the rationality of the quotient threefold of a 3-dimensional torus of product type.

1. Introduction

Unless stated otherwise, we work in the category of projective varieties defined over the complex number field **C**. Our main result is Theorem 1.5 below. We first introduce some background for the study in this paper.

1.1. **General Problem.** Complex dynamics of biholomorphic automorphisms of compact Kähler surfaces is now fairly well-understood since Cantat [Ca99]. Especially, Bedford-Kim ([BK09], [BK12], [BK10]) and McMullen ([Mc02], [Mc07], [Mc11], [Mc13]) show very beautiful aspects of automorphisms of rational surfaces and K3 surfaces.

On the other hand, not much is known in higher dimensions. Especially, the following one of the most basic problems is completely open:

Problem 1.1. Find (many) examples of rational manifolds and Calabi-Yau manifolds admitting *primitive* biregular automorphisms of *positive entropy.*

Remark 1.2. (1) Here a birational selfmap f of a manifold M is *imprimitive* if f comes from lower dimensional manifolds, or more precisely, there are a dominant rational map $\varphi : M \cdots \to B$ with $0 < \dim B < \dim M$ and a dominant rational map $f_B : B \cdots \to B$ such that $\varphi \circ f = f_B \circ \varphi$. A birational selfmap that is not imprimitive is *primitive.* This notion was introduced by Zhang [Zh09] and Nakayama-Zhang [NZ09].

(2) *Entropy* is an important invariant that measures how fast two general points spread out under the action of $\langle f \rangle$. The original definition is a completely topological one, involving a metric $d(x, y)$ on the manifold and the induced metrics $d_n(x, y) = \max_{0 \le j \le n} d(f^j x, f^j y)$ (See e.g. Bowen [Bo73] for the case of continuous maps, and for the case of rational maps see Friedland [Fr91] and Guedj [Gu05].) When f is a biregular automorphism of a compact

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Kähler manifold, a fundamental theorem of Gromov [Gr03] and Yomdin [Yo87] says that the entropy of f is

$$h_{top}(f) = \log \max_{0\le k\le \dim M} d_k(f) \ ,$$

where $d_k(f)$ is the k-th *dynamical degree*, i.e., the spectral radius of $f^*|H^{2k}(M, \mathbf{Z})$. In particular, if f is of positive entropy, then f is of infinite order. The theorem of Gromov and Yomdin was partially extended to the case of meromorphic maps in Dinh-Sibony [DS05][DS04].

The main aim of this note is to present first explicit examples of a rational threefold and a Calabi-Yau threefold in Problem 1.1.

1.2. **Constructing rational threefolds with rich automorphisms.** The most essential part of Theorem 1.5 is the rationality of the obtained threefold. To explain this, we recall two most basic constructions of manifolds from an existing manifold V:

(I) Take a finite successive blow up M of V along smooth centers.

(II) Take a nice resolution M of singularities of the quotient variety V/G by a finite subgroup $G \subset \mathrm{Aut}\,(V)$.

In construction (I), it is clear that M is rational if $V = \mathbf{P}^3$, and in this way, there are constructed some explicit, interesting examples of *birational* automorphisms of rational threefolds of infinite orders (Bedford-Kim [BK11][BK13], Perroni-Zhang [PZ11], Blanc [Bl12]). However, they are either only pseudo-automorphisms (i.e., birational selfmaps isomorphic in codimension one) but not biregular, or imprimitive, or of null-entropy. Moreover, in all of these examples the first dynamical degrees are either 1 or a Salem number, and hence by the results in [OT13] the first and second dynamical degrees of these examples must be the same (compare with our construction in Section 3 and Corollary 3.2).

In fact, in dimension ≥ 3, it is rather hard to construct *biregular* automorphisms, compared with finding birational automorphisms or pseudo-automorphisms. For instance, the following very simple question by Professor Eric Bedford (at a conference in Paris in 2011), which was studied by the second author in [Tr12], still remains open:

Question 1.3. Is there a smooth rational threefold W obtained by a successive blow-up of $\mathbf{P}^3$ along smooth centers such that W admits biregular automorphisms of positive entropy?

The results in [Tr12] suggest that the answer to Question 1.3 could be negative. So, at the moment, construction (I) may not be so promising to Problem 1.1.

In the second construction (II), M has many biregular automorphisms if V has many biregular automorphisms normalizing G. However the rationality of V/G is highly nontrivial (even if V itself is rational!). There are now many general methods to see if M is uniruled, rationally connected or not ([MM86], [KMM92], [KL09]), and some useful methods to conclude M is *not* rational ([IM71], [CG72], [AM72], [Ko95], [Be12]). However, in dimension ≥ 3, there is essentially no general method to conclude M is rational or unirational, and this is in general very hard ([Ko02], see also the excellent survey [Ko01]). For instance, the following Question 1.4, which is closely related to our construction, still remains open ([Ca12]), while Campana himself showed that Z in Question 1.4 is certainly rationally connected:

Question 1.4. Let $E_{\sqrt{-1}} := \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\sqrt{-1})$ be the elliptic curve of period $\sqrt{-1}$. Let Z be the canonical resolution ([Ue75, Page 199]) of the quotient threefold

$$E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}}/\langle \mathrm{diag}\,(\sqrt{-1}, \sqrt{-1}, \sqrt{-1})\rangle\ ,$$

i.e., the blow up at the singular points of type $1/2(1, 1, 1)$ and $1/4(1, 1, 1)$. Is this Z rational or unirational?

1.3. **Main Result.** Let

$$\omega := \frac{-1 + \sqrt{-3}}{2}$$

and

$$E_\omega := \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\omega)$$

be the elliptic curve of period ω. Similar to $E_{\sqrt{-1}}$, the elliptic curve E_ω is a very special one that is characterized as the unique elliptic curve admitting an automorphism c of order 3 such that $c^*\sigma = \omega\sigma$. Here σ is a non-zero holomorphic 1-form on the elliptic curve.

Let X be the canonical resolution of the quotient threefold

$$E_\omega \times E_\omega \times E_\omega/\langle \mathrm{diag}\,(\omega, \omega, \omega)\rangle\ ,$$

i.e., the blow-up at the singular points of type $1/3(1, 1, 1)$ and Y be the canonical resolution of the quotient threefold

$$E_\omega \times E_\omega \times E_\omega/\langle \mathrm{diag}\,(-\omega, -\omega, -\omega)\rangle\ ,$$

i.e., the blow-up at the singular points of type $1/2(1, 1, 1)$, $1/3(1, 1, 1)$, $1/6(1, 1, 1)$.

Y is then the canonical resolution of X/ι, where ι is the involution naturally induced by $\mathrm{diag}\,(-1, -1, -1)$. It is well-known that X is a Calabi-Yau threefold with very special properties, e.g., it is rigid and plays a crucial role in the classification of fiber space structures on Calabi-Yau threefolds ([Be82], [OS01] and references therein). Our theorem shows that X has also a rich structure in the complex dynamical view. We remark that also for Calabi-Yau manifolds, it seems much harder to construct biregular automorphisms compared with birational selfmaps (See for instance [Og12], [CO12]). On the other hand, our new manifold Y seems so far caught no attention.

Now we state the main result:

Theorem 1.5. *(1) Y is rational, i.e., birationally equivalent to $\mathbf{P}^3$.*

(2) Both X and Y admit primitive biregular automorphisms of positive entropy whose first dynamical degree is not a Salem number.

Our X and Y are the first explicit examples of Calabi-Yau threefolds and rational threefolds which admit biregular, primitive automorphisms of positive entropy. As we explained in the last subsection, the crucial point of the proof is the rationality of Y. Our method is quite elementary but tricky. In the course of proof, we also find a birational model of Y, which is a quintic hypersurface of very simple form (Theorem 2.5). This birational model and its rational conic bundle structure are important to conclude the rationality. It would be interesting to see if Z in Question 1.4 is rational or not also in this view.

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2. Rationality of Y

In this section, we shall show that Y in Theorem 1.5 is rational.

Lemma 2.1. *E_ω is the projective non-singular model, say $\tilde{C}$, of the affine curve*

$$C : y^2 = x^3 - 1 \ .$$

Moreover, the complex multiplication $-\omega$ on E_ω is the extension $\tilde{g}$ of the automorphism of C given by

$$g^*(x, y) \mapsto (\omega x, -y) \ .$$

Proof. The projective curve $y^2 t = x^3 - t^3$ is a smooth cubic in $\mathbf{P}^2$ with homogeneous coordinates $[t : x : y]$, hence it defines an elliptic curve $\tilde{C}$ which is the compactification of C in $\mathbf{P}^2$. The rational 1-form dx/y on C defines a regular 1-form σ of $\tilde{C}$. Since $g^*dx/y = -\omega dx/y$ and the point $O = [0 : 0 : 1]$ is a fixed point, the unique extension $\tilde{g}$ of g defines an automorphism of $\tilde{C}$ such that $\tilde{g}^*\sigma = -\omega\sigma$ with a fixed point O. Thus, identifying the origin of E_ω with the point O, we obtain $(E_\omega, -\omega) \simeq (\tilde{C}, \tilde{g})$. □

Let $k \in \{1, 2, 3\}$ and let $\mathbf{C}^2_k$ be the affine plane with coordinates (x_k, y_k), and

$$C_k : y_k^2 = x_k^3 - 1 \ , \ g_k : (x_k, y_k) \mapsto (\omega x_k, -y_k) \ ,$$

$$V := C_1 \times C_2 \times C_3 \ , \ g = g_1 \times g_2 \times g_3 \ .$$

By Lemma 2.1, our Y is birationally equivalent to the affine threefold $W := V/\langle g\rangle$.

Lemma 2.2. *The affine coordinate ring $\mathbf{C}[W]$ of W is the subring*

$$R := \mathbf{C}[y_1^2, y_2^2, y_3^2, y_1y_2, y_2y_3, y_3y_1, x_1^i x_2^j x_3^k \, (0 \le i, j, k \le 2 \, , \, i + j + k = 3)] \ ,$$

of the quotient ring

$$\mathbf{C}[y_1, y_2, y_3, x_1, x_2, x_3]/(y_1^2 - x_1^3 + 1, y_2^2 - x_2^3 + 1, y_3^2 - x_3^3 + 1) \ .$$

Proof. Since $V = C_1 \times C_2 \times C_3$, the affine coordinate ring of V is

$$\mathbf{C}[y_1, x_1]/(y_1^2 - x_1^3 + 1) \otimes_{\mathbf{C}} \mathbf{C}[y_2, x_2]/(y_2^2 - x_2^3 + 1) \otimes_{\mathbf{C}} \mathbf{C}[y_3, x_3]/(y_3^2 - x_3^3 + 1) \ .$$

This ring is naturally isomorphic to

$$\tilde{R} := \mathbf{C}[y_1, y_2, y_3, x_1, x_2, x_3]/(y_1^2 - x_1^3 + 1, y_2^2 - x_2^3 + 1, y_3^2 - x_3^3 + 1) \ .$$

The action of g on $\tilde{R}$ is given by $gy_k = -y_k$ and $gx_k = \omega x_k$ and the affine coordinate ring of $V/\langle g\rangle$ is isomorphic to the invariant ring $\tilde{R}^g$. Here and hereafter, we denote the action of g on $\tilde{R}$, which is g^*, simply by g.

By the shape of $\tilde{R}$, each element of $\tilde{R}$ is uniquely expressed in the following form:

$$f := \sum_{(i,j,k) \in \{0,1,2\}^3} a_{ijk} x_1^i x_2^j x_3^k$$

where $a_{ijk} = a_{ijk}(y_1, y_2, y_3)$ are polynomials of y_1, y_2, y_3. Since $\langle g\rangle = \langle g^2, g^3\rangle$, we have

$$\tilde{R}^g = (\tilde{R}^{g^2})^{g^3} \ .$$

Note that $g^2 y_m = y_m$ and $g^2 x_m = \omega^2 x_m$ $(m = 1, 2, 3)$. Hence, any polynomial $a_{ijk}(y_1, y_2, y_3)$ are g^2-invariants and therefore $f \in R = \tilde{R}^{g^2}$ if and only if $x_1^i x_2^j x_3^k$ are all g^2-invariant (for $a_{i,j,k} \neq 0$). That is, $i + j + k$ is divisible by 3:

$$3|(i + j + k) \ .$$

Note then that $i + j + k = 0$, 3 and 6 and the term of $i + j + k = 6$ is only $(x_1 x_2 x_3)^2$ by $0 \le i, j, k \le 2$. Hence

$$\tilde{R}^{g^2} = \mathbf{C}[y_1, y_2, y_3, x_1^i x_2^j x_3^k \, (0 \le i, j, k \le 2 \ , \ i + j + k = 3)] \ .$$

Since $g^3 x_m = x_m$ and $g^3 y_m = -y_m$ $(m = 1, 2, 3)$, it follows that any polynomial in x_1, x_2, x_3 is g^3- invariant. Hence f is g^3-invariant if and only if for any monomial $y_1^{i_1} y_2^{i_2} y_3^{i_3}$ appearing in a_{ijk} we have $i_1 + i_2 + i_3 =$ an even number. Therefore

$$(\tilde{R}^{g^2})^{g^3} = \mathbf{C}[y_m y_n \, (1 \le m \le n \le 3) \, , \, x_1^i x_2^j x_3^k \, (0 \le i, j, k \le 2 \ , \ i + j + k = 3)] \ ,$$

as claimed. □

Lemma 2.3. *The rational function field* $\mathbf{C}(Y)$ *of* Y*, hence the function field of* W*, is isomorphic to*

$$\mathbf{C}(y_1^2, \frac{y_2}{y_1}, \frac{y_3}{y_1}, \frac{x_2}{x_1}, \frac{x_3}{x_1}) \ .$$

Proof. It is clear that the field above, say K, is a subfield of $\mathbf{C}(W)$ (e.g. $y_2/y_1 = (y_2^2)/(y_1 y_2)$ and $x_2/x_1 = (x_2^2 x_1)/(x_1^2 x_2)$). It suffices to show that each generator of $\mathbf{C}[W]$ in Lemma 2.2 is in K. For $y_m y_n$, we have

$$y_m y_n = y_1^2 \frac{y_m}{y_1} \frac{y_n}{y_1} \ \in K \ .$$

For $x_1^i x_2^j x_3^k$ with $i + j + k = 3$, we have

$$x_1^i x_2^j x_3^k = (\frac{x_2}{x_1})^i (\frac{x_3}{x_1})^j (x_1)^3 = (\frac{x_2}{x_1})^i (\frac{x_3}{x_1})^j (y_1^2 + 1) \ \in K \ .$$

Hence the result follows. □

Lemma 2.4. *The rational function field* $\mathbf{C}(Y)$ *of* Y *is isomorphic to*

$$\mathbf{C}(t, s, z, w) \ ,$$

with a single equation

$$(w^3 - 1)(t^2 - 1) = (z^3 - 1)(s^2 - 1) \ .$$

More precisely, $\mathbf{C}(Y)$ *is isomorphic to*

$$\mathbf{C}(s, z, w)[T]/I$$

where I *is the principal ideal generated by*

$$(w^3 - 1)(T^2 - 1) - (z^3 - 1)(s^2 - 1) \ ,$$

in the polynomial ring $\mathbf{C}(s, z, w)[T]$ *over* $\mathbf{C}(s, z, w)$.

Proof. Set

$$(1)\ \ u := y_1^2\ ,\ t := \frac{y_2}{y_1}\ ,\ s := \frac{y_3}{y_1}\ z := \frac{x_2}{x_1}\ ,\ w := \frac{x_3}{x_1}\ .$$

These 5 elements are generators of $\mathbf{C}(W) = \mathbf{C}(Y)$. Then

$$(2)\ \ y_2 = ty_1\ ,\ y_3 = sy_1\ ,\ y_2^2 = t^2u\ ,\ y_3^2 = s^2u\ .$$

Hence

$$(3)\ \ z^3 = \frac{x_2^3}{x_1^3} = \frac{y_2^2+1}{y_1^2+1} = \frac{t^2u+1}{u+1}\ ,\ w^3 = \frac{x_3^3}{x_1^3} = \frac{y_3^2+1}{y_1^2+1} = \frac{s^2u+1}{u+1}\ .$$

Solving each equation in (3) in the variable u, we obtain

$$u = \frac{z^3-1}{t^2-z^3}\ ,\ u = \frac{w^3-1}{s^2-z^3}$$

that is,

$$(4)\ \ \frac{-1}{u} = \frac{z^3-t^2}{z^3-1} = 1 - \frac{t^2-1}{z^3-1}$$

and

$$(5)\ \ \frac{-1}{u} = \frac{w^3-ts^2}{w^3-1} = 1 - \frac{s^2-1}{w^3-1}\ .$$

Hence by (4) and (5), we find that $\mathbf{C}(W) = \mathbf{C}(t,s,z,w)$ and z, w, s, t satisfy a relation

$$\frac{t^2-1}{z^3-1} = \frac{s^2-1}{w^3-1}$$

that is

$$(6)\ \ (w^3-1)(t^2-1) = (z^3-1)(s^2-1)\ .$$

Since W is a 3-dimensional projective variety, it follows from (6) that s, z, w are transcendental basis of $\mathbf{C}(W)$ and t is algebraic over the field $\mathbf{C}(s,z,w)$. Therefore

$$\mathbf{C}(W) = \mathbf{C}(s,z,w)(t) = \mathbf{C}(s,z,w)[t]\ .$$

The equation (6), or more precisely, the polynomial

$$(7)\ \ (w^3-1)(T^2-1) - (z^3-1)(s^2-1)$$

gives an algebraic equation of t over $\mathbf{C}(s,z,w)$. Since the polynomial (7) is irreducible over $\mathbf{C}(s,z,w)$, it follows that

$$\mathbf{C}(s,z,w)[t] = \mathbf{C}(s,z,w)[T]/I$$

as claimed. □

Theorem 2.5. *Y is birationally equivalent to the hypersurface Q defined by*

$$(x_1^3-x_0^3)(x_3^2-x_0^2) = (x_2^3-x_0^3)(x_4^2-x_0^2)$$

in $\mathbf{P}^4$ with homogeneous coordinates $[x_0 : x_1 : x_2 : x_3 : x_4]$.

Proof. By Lemma 2.4, Y is birationally equivalent to the affine hypersurface defined by

$$(w^3-1)(t^2-1) = (z^3-1)(s^2-1)$$

in $\mathbf{C}^4$ with affine coordinates (w,z,t,s), and the result follows. □

The next proposition completes the proof of Theorem 1.5 (1):

Proposition 2.6. *The hypersurface Q in Theorem 2.5 is rational.*

Proof. Let $L \subset \mathbf{P}^4$ be the projective line defined by

$$x_0 = x_1 = x_2 = 0 \ .$$

Note that $L \subset Q$. Consider the linear projections from L:

$$\pi : \mathbf{P}^4 \cdots \to P := \mathbf{P}^2$$

$$\pi_Q : Q \cdots \to P = \mathbf{P}^2$$

where $P = \mathbf{P}^2$ is defined by $x_3 = x_4 = 0$ in $\mathbf{P}^4$ with homogeneous coordinates $[x_0 : x_1 : x_2 : 0 : 0]$. Let $b := x_1/x_0$, $c := x_2/x_0$. Then $\mathbf{C}(P) = \mathbf{C}(b, c)$ and the point $\eta = [1 : b : c : 0 : 0]$ of P is the generic point of P in the sense of scheme. Then the fiber $\pi^{-1}(\eta)$ is the plane $\mathbf{P}^2_\eta$ over $\mathbf{C}(P)$ consisting of the forms:

$$(*) \ \ [\alpha : \alpha b : \alpha c : \beta : \gamma] \ ,$$

where $[\alpha : \beta : \gamma]$ is regarded as a linear homogeneous coordinates of the plane $\pi^{-1}(\eta) = \mathbf{P}^2_\eta$ over $\mathbf{C}(P)$. Hence, by substituting (*) into the equation of Q and simplifying it by dividing out by α^3, we find that the fiber $C_\eta := \pi_Q^{-1}(\eta)$ is the conic defined by

$$(b^3 - 1)(\beta^2 - \alpha^2) = (c^3 - 1)(\gamma^2 - \alpha^2)$$

in $\pi^{-1}(\eta) = \mathbf{P}^2_\eta$ over $\mathbf{C}(P)$. Hence π_Q is a rational conic bundle over $P = \mathbf{P}^2$. Moreover, the point

$$[\alpha : \beta : \gamma] = [1 : 1 : 1]$$

is clearly a rational point of C_η also over $\mathbf{C}(P)$. Hence $C_\eta \simeq \mathbf{P}^1_\eta$ over $\mathbf{C}(P)$. For this one may just consider the projection from the above rational point over $\mathbf{C}(P)$. Note that the generic point $\tilde{\eta}$ of Q is lying over η. Denote the residue field of $\tilde{\eta}$ (resp. of η) by $K(\tilde{\eta})$ (resp. $K(\eta)$). Then $K(\eta) = \mathbf{C}(P) = \mathbf{C}(b, c)$, $K(\eta) \subset K(\tilde{\eta})$ and

$$\mathbf{C}(Q) = K(\tilde{\eta}) = K(\eta)(C_\eta) \simeq K(\eta)(\mathbf{P}^1_\eta) = \mathbf{C}(b, c)(t) = \mathbf{C}(b, c, t) \ ,$$

where t is the affine coordinate function of $\mathbf{P}^1_\eta$. Since $\dim Q = 3$, the rational functions b, c and t are algebraically independent over $\mathbf{C}$. Hence $\mathbf{C}(Q)$ is purely transcendental over $\mathbf{C}$. This implies the result. □

Remark 2.7. $\tilde{C}$ in Lemma 2.1 is the elliptic curve defined over any field K of characteristic $\neq 2, 3$ and the automorphism $\tilde{g}$ is defined over any field K containing the primitive third root of unity ω. The argument in this section shows that $V/\langle g \rangle$ is birationally equivalent to Q and is rational, also over any field K containing ω and of characteristic $\neq$ 2, 3.

Remark 2.8. Consider the quotient variety of dimension $n \geq 2$:

$$V_n := E^n_\omega / \langle -\omega I_n \rangle \ .$$

Then V_n has isolated singular points of type $1/2(1, 1, \cdots, 1)$, $1/3(1, 1, \cdots, 1)$ and one isolated singular point O of type $1/6(1, 1, \cdots, 1)$. From this, it is easy to see that $\mathcal{O}_{V_n}(6K_{V_n}) \simeq \mathcal{O}_{V_n}$, $h^1(\mathcal{O}_{V_n}) = 0$ and the singular point O is Kawamata log terminal but not canonical when $2 \leq n \leq 5$, canonical but not terminal when $n = 6$ and terminal (but not smooth) when $n \geq 7$. Also all other singular points are terminal when $n \geq 4$. (See [KMM87], [KM98] for terminologies and basic notions of minimal model theory.) So, when $n \geq 7$

(resp. when $n = 6$), V_n (resp. the blow up of V_6 at O) are minimal but singular Calabi-Yau varieties, in the sense of minimal model theory, and therefore, they are not even uniruled. On the other hand, the Kodaira dimension of the resolution of V_n ($2 \le n \le 5$) is $-\infty$. So, V_2 is rational by Castelnouvo's criterion (see eg. [BHPV04, Page 252]). Our result shows that V_3 is rational. It would be interesting to see if V_4, V_5 are rational, unirational or not.

3. Primitive automorphisms of X and Y

In this section, we prove Theorem 1.5 (2).

Dynamical degrees of dominant meromorphic selfmaps of compact Kähler manifolds were defined by Dinh-Sibony [DS05][DS04], who proved that dynamical degrees are bimeromorphic invariants. Relative dynamical degrees were introduced by Dinh-Nguyen [DN11], who used these to establish some relations between dynamical degrees of dominant rational maps of projective manifolds which are semi-conjugate to each other. (These relations were extended to the case of compact Kähler manifolds in [DNT11], see also [NZ09] for related results). The following theorem is proved in [DN11, Theorem 1.1, Corollary 1.2] (see also Section 3 of [DN11] for the definition of the relative dynamical degrees $d_k(f|\pi)$):

Theorem 3.1. *Let M, B be projective manifolds of dimension k and l with $k \ge l$. Let $f : M \cdots \to M$, $f_B : B \cdots \to B$ and $\pi : M \cdots \to B$ be dominant rational maps such that $\pi \circ f = f_B \circ \pi$. Then*

(1) $d_p(f) \ge 1$ ($0 \le p \le k$), $d_j(f_B) \ge 1$ ($0 \le j \le l$), $d_{p-j}(f|\pi) \ge 1$ ($0 \le p \le k-l$), $d_0(f|\pi) = 1$ and

$$d_p(f) = \mathrm{Max}_{j \in J} d_j(f_B) d_{p-j}(f|\pi)$$

where $J := \{j \in \mathbf{Z} \,|\, 0 \le j \le \ell, 0 \le p-j \le k-l\}$, that is, J is the set of j such that both $d_j(f_B)$ and $d_{p-j}(f|\pi)$ make a sense.

(2) In particular, if $k = l$, i.e., if $\pi : M \cdots \to B$ is a generically finite rational map, then the dynamical degrees of f and f_B are the same, i.e., $d_p(f) = d_p(f_B)$ for all integers p with $0 \le p \le k = l$.

Applying for threefolds, we obtain the following strong restriction of the dynamical degrees of imprimitive birational selfmaps of threefolds:

Corollary 3.2. *Let M be a smooth projective threefold and f be a birational selfmap of M. Assume that f is imprimitive, i.e., there are a smooth projective manifold B of $0 < \dim B < 3$, a birational selfmap $f_B : B \cdots \to B$ of B and a dominant rational map $\pi : M \cdots \to B$ such that $\pi \circ f = f_B \circ \pi$. Then $d_1(f) = d_2(f)$. In other words, if $d_1(f) \ne d_2(f)$, then f has to be primitive.*

Proof. Since f is a birational map of a threefold, it follows that $d_0(f) = d_3(f) = 1$. By Theorem 3.1(1), $d_3(f) = d_2(f_B)d_1(f|\pi)$ when $\dim B = 2$ and $d_3(f) = d_1(f_B)d_2(f|\pi)$ when $\dim B = 1$. Therefore, $d_2(f_B) = d_1(f|\pi) = 1$ when $\dim B = 2$ and $d_1(f_B) = d_2(f|\pi) = 1$ when $\dim B = 1$, again by Theorem 3.1(1). Hence when $\dim B = 2$, it follows once again from Theorem 3.1(1) that $d_1(f) = d_1(f_B) = d_2(f)$. Similarly, $d_1(f) = d_1(f|\pi) = d_2(f)$ when $\dim B = 1$. □

Set $M := E_\omega \times E_\omega \times E_\omega$. Then we have a natural embedding of groups

$$\mathrm{GL}\,(3, \mathbf{Z}[\omega]) \subset \mathrm{Aut}\,(M)\ .$$

Moreover, since $\pm\mathrm{diag}\,(\omega,\omega,\omega)$ is in the center of $\mathrm{GL}(3,\mathbf{Z}[\omega])$, it follows that any $f \in \mathrm{GL}(3,\mathbf{Z}[\omega])$ naturally descends to the biregular automorphisms of X and Y. We denote them by f_X and f_Y respectively. The regularity of f_X and f_Y is a consequence of the universal property of blow-up.

Lemma 3.3. *Let a be any positive integer. Consider the automorphism $f := f_a$ of M given by the matrix*

$$P = P_a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 3a^2 & 0 \end{pmatrix} .$$

Then $d_2(f) > d_1(f) > 1$. Moreover $d_1(f)$ is not a Salem number.

Note here that a *Salem number* is a real algebraic integer $a > 1$ with Galois conjugates a, $1/a$ such that all other Galois conjugates are on the unit circle $S^1 \subset \mathbf{C}$.

Proof. Note that $P \in \mathrm{SL}(3,\mathbf{Z})$ so that P gives an automorphism of M. The characteristic polynomial $\Phi(t)$ of the matrix P is

$$\Phi(x) = x^3 - 3a^2x + 1 .$$

Observe $\Phi'(x) = 3(x-a)(x+a)$, and

$$\Phi(-a) = 2a^2 + 1 > 0,\ \Phi(0) = 1 > 0,\ \Phi(1) = 2 - 3a^2 < 0,\ \Phi(a) = -2a^2 + 1 < 0 .$$

Hence $\Phi(x)$ has three real roots α β, γ such that

$$\alpha < -a < 0 < \beta < 1 < a < \gamma .$$

We have also

$$\alpha + \beta + \gamma = 0$$

by the shape of $\Phi(x)$. Hence

$$0 < |\beta| < 1 < |\gamma| < |\alpha| .$$

Observe that the characteristic polynomial of $f^*|H^1(M,\mathbf{Z})$ is $\Phi(x)^2$. Hence α, β, γ are the eigenvalues of $f^*|H^1(M,\mathbf{Z})$ each of which is of multiplicity 2. Since

$$H^{2k}(M,\mathbf{Z}) = \wedge^{2k}H^1(M,\mathbf{Z}) ,$$

it now follows that $d_1(f) = \alpha^2$ and $d_2(f) = \alpha^2\gamma^2$, therefore $d_2(f) > d_1(f) > 1$.

Since f is an automorphism in dimension 3 and $d_1(f) \neq d_2(f)$, by the results in [OT13] we have that $d_1(f)$ is not a Salem number. We can also see this directly as follows. Note that $\Phi(x)$ is irreducible over $\mathbf{Q}$, or equivalently, over $\mathbf{Z}$. Indeed, otherwise, $\Phi(x)$ would have one of ± 1 as its root, a contradiction. Hence the three roots α, β and γ are Galois conjugate over $\mathbf{Q}$. Thus, so are α^2, β^2, γ^2. Since $\alpha^2 > \gamma^2 > 1$, it follows that $d_1(f) = \alpha^2$ is not a Salem number. □

The next proposition completes the proof of Theorem 1.5 (2):

Proposition 3.4. *The induced automorphisms f_X and f_Y are primitive and of positive entropy for $f := f_a$ in Lemma 3.3. Moreover, their first dynamical degrees are not Salem numbers.*

Proof. By Theorem 3.1(2), we have

$$d_p(f_X) = d_p(f) = d_p(f_Y)\ .$$

By Lemma 3.3, we have $d_2(f) > d_1(f)$. Hence

$$d_2(f_X) > d_1(f_X) > 1\ ,\ d_2(f_Y) > d_1(f_Y) > 1\ .$$

Thus f_X and f_Y are primitive by Corolary 3.2 and of positive entropy by $d_1(f_X) > 1$ and $d_1(f_Y) > 1$. By Lemma 3.3, $d_1(f_X) = d_1(f_Y) = d_1(f)$ is not a Salem number as well. □

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