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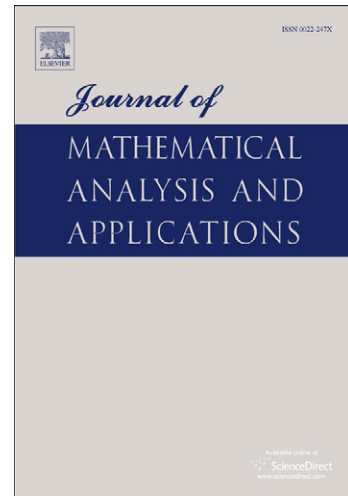
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SETS NON-THIN AT ∞ IN \mathbb{C}^m

TUYEN TRUNG TRUONG

ABSTRACT. In this paper we define the notion of non-thin at ∞ as follows: Let E be a subset of \mathbb{C}^m . For any $R > 0$ define $E_R = E \cap \{z \in \mathbb{C}^m : |z| \leq R\}$. We say that E is non-thin at ∞ if

$$\lim_{R \rightarrow \infty} V_{E_R}(z) = 0$$

for all $z \in \mathbb{C}^m$, where V_E is the pluricomplex Green function of E .

This definition of non-thinness at ∞ has good properties: If $E \subset \mathbb{C}^m$ is non-thin at ∞ and A is pluripolar then $E \setminus A$ is non-thin at ∞ ; if $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$ are arbitrary sets, then E and F are non-thin at ∞ iff $E \times F \subset \mathbb{C}^m \times \mathbb{C}^n$ is non-thin at ∞ (see Lemma 2).

The results of this paper extend some results in [8] and [3].

1. INTRODUCTION

Fix $m \in \mathbb{N}$, let \mathbb{C}^m be the usual m -dimensional complex Euclidean space. Before going into the main points, we recall some facts about the potential theory in \mathbb{C}^m . Let U be an open subset of \mathbb{C}^m . A function $u : U \rightarrow [-\infty, \infty)$ is called PSH in U (written $u \in PSH(U)$) if u is upper-semicontinuous and when restricted to any complex line $L \simeq \mathbb{C}$ then u is subharmonic (see [4]).

A $PSH(\mathbb{C}^m)$ function u is said to be of minimal growth if

$$u(z) - \log |z| \leq O(1), \text{ as } |z| \rightarrow \infty,$$

here $|z|$ is the usual Euclidean norm of an element $z \in \mathbb{C}^m$. We denote the set of all such functions by \mathbb{L} .

Let E be a subset in \mathbb{C}^m . Then the pluricomplex Green function of the set E with pole at infinity (see [4], page 184) is

$$V_E(z) = \sup\{u(z) : u \in \mathbb{L}, u|_E \leq 0\} \quad (z \in \mathbb{C}^m).$$

V_E is also called the Siciak extremal function of the set E . By definition we see that $V_E \geq 0$.

For any function $u : \mathbb{C}^m \rightarrow [-\infty, \infty)$ we define its upper-semicontinuous regularization u^* (see [4]) by

$$u^*(z) = \limsup_{\zeta \rightarrow z} u(\zeta).$$

Let $V_E^*(z)$ be the regularization of the Green function $V_E(z)$. It is well-known (see [4]) that there are only three cases:

Case 1: $V_E^*(z) \equiv +\infty$. Then E is pluripolar.

Case 2: $V_E^*(z) \in \mathbb{L}$, and $V_E^* \not\equiv 0$.

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Case 3: $V_E^*(z) \equiv 0$.

When $m = 1$ then $V_E^*(z) \equiv 0$ iff the set E is non-thin at ∞ , or equivalently the set $E^* = \{z : 1/z \in E\}$ is non-thin at 0 (see [8], [2]). Recall that a subset E of \mathbb{C}^m is pluri-thin (or thin for brevity) at a point $a \in \mathbb{C}^m$ if either a is not a limit point of E or there is a neighborhood U of a and a function $u \in PSH(U)$ such that

$$\limsup_{z \rightarrow a, z \in E \setminus a} u(z) < u(a).$$

If E is not thin then it is called non-thin.

In [8] the authors proved the following result (see also [2], page 270)

Proposition 1. *Let E be a closed subset of \mathbb{C} . Then the following four statements are equivalent*

1. E is non-thin at ∞ .
2. For any $z \in \mathbb{C}$ we have

$$\lim_{R \rightarrow \infty} V_{E_R}(z) = 0,$$

where we define $E_R = E \cap \{z \in \mathbb{C} : |z| \leq R\}$.

3. If sequences (P_n) of polynomials and $k_n \geq \deg(P_n)$ satisfy

$$\limsup_{n \rightarrow \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in E$, then

$$\limsup_{n \rightarrow \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in \mathbb{C}$.

4. $V_E(z) \equiv 0$.

In \mathbb{C}^m ($m > 1$), we can not reduce the definition of non-thinness of E at ∞ to the non-thinness at 0 of some other sets E^* , as it was when $m = 1$. However, Proposition 1 suggests a way to define non-thinness at ∞ in higher dimensions:

Definition 1. *Let E be a subset of \mathbb{C}^m . For any $R > 0$ define $E_R = E \cap \{z \in \mathbb{C}^m : |z| \leq R\}$. We say that E is non-thin at ∞ if*

$$\lim_{R \rightarrow \infty} V_{E_R}(z) = 0$$

for all $z \in \mathbb{C}^m$.

If E is not non-thin at ∞ then we say E is thin at ∞ .

This definition of non-thinness at ∞ has good properties: If $E \subset \mathbb{C}^m$ is non-thin at ∞ and A is pluripolar then $E \setminus A$ is non-thin at ∞ ; if $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$ are arbitrary sets, then E and F are non-thin at ∞ iff $E \times F \subset \mathbb{C}^m \times \mathbb{C}^n$ is non-thin at ∞ (see Lemma 2).

Our first result in this paper is the following, which is an analog to the case $m = 1$:

Theorem 2. *Let E be a closed subset of \mathbb{C}^m . Then the following two statements are equivalent:*

1. E is non-thin at ∞ .
2. If sequences (P_n) of polynomials and $k_n \geq \deg(P_n)$ satisfy

$$\limsup_{n \rightarrow \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in E$, then

$$\limsup_{n \rightarrow \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in \mathbb{C}^m$.

If E is non-thin at ∞ then it is easy to see that $V_E \equiv 0$. The converse is not true for $m \geq 2$, by the following example (which essentially is the same as Example 1.1 in [2]).

Example 1. Let E be a subset of \mathbb{C}^2 defined by

$$E = \{(z_1, z_2) \in \mathbb{C}^2 : |z_2| \leq 1\} \cup \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = 0\}.$$

Then $V_E \equiv 0$ but E is thin at ∞ .

Proof. Use arguments in Example 1.1 in [2], it is easy to see that $V_E \equiv 0$.

If E was non-thin at ∞ then since $A = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = 0\}$ is pluripolar, by Lemma 2 we should have $E \setminus A$ is non-thin at ∞ . In particular we should have $V_{E \setminus A} \equiv 0$. However as computed in [2] we have $V_{E \setminus A}(z) = \log^+ |z_2|$, which is a contradiction.

Hence E is thin at ∞ . \square

The following result gives a characterization of sets E with $V_E \equiv 0$, which also shows that the non-thin at ∞ sets are not rare.

Theorem 3. Let E be a closed subset of \mathbb{C}^m . Then $V_E \equiv 0$ iff any open neighborhood of E is non-thin at ∞ .

Example 2. Applying Theorem 3 to Example 1, we see that the set

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_2| \leq 1\} \cup W,$$

is non-thin at ∞ in \mathbb{C}^2 , where W is an open neighborhood of the set $\{(z_1, z_2) \in \mathbb{C}^2 : z_1 = 0\}$.

Example 3. Let Δ be a collection of complex lines L in \mathbb{C}^m such that

$$\bigcup_{L \in \Delta} L = \mathbb{C}^m.$$

Let E be a closed subset of \mathbb{C}^m such that for each $L \in \Delta$ the set $E \cap L$ considered as a subset in the one-dimensional complex line L is non-thin at ∞ . Then E is non-thin at ∞ as a subset in \mathbb{C}^m .

Proof. By Theorem 2 we need only to show that: If (P_n) is a sequence of polynomials, and $k_n \geq \deg(P_n)$ such that

$$\limsup_{n \rightarrow \infty} |P_n(z)|^{1/k_n} \leq 1$$

for $z \in E$, then

$$\limsup_{n \rightarrow \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in \mathbb{C}^m$.

Let $w \in \mathbb{C}$ be a coordinate for L such that the coordinates z_1, \dots, z_n of \mathbb{C}^m are linear functions of w when restricted on L , and denote by $P_{n,L}(w)$ the restriction of P_n to L . Then $P_{n,L}(w)$ is a polynomial in one complex variable w of degree $\deg(P_{n,L}(w)) \leq \deg(P_n) \leq k_n$.

Then since

$$\limsup_{n \rightarrow \infty} |P_{n,L}(w)|^{1/k_n} \leq 1$$

for $w \in E \cap L$, and $E \cap L$ is non-thin at ∞ in the complex line L , hence

$$\limsup_{n \rightarrow \infty} |P_n(w)|^{1/k_n} \leq 1$$

for all $w \in L$. Since this is true for any complex line $L \in \Delta$, it is also true for their union, which is equal to \mathbb{C}^m . \square

In Theorem 2 of [3], we prove a Phragmen-Lindelof type inequality for subsets of \mathbb{C} having "big" capacity (by "big" capacity we mean a set E for which $C(E_R)$ grows proportionally to $\log R$). In the sequel, we state and prove a similar result for higher dimensions (see the remarks at the end of this section for more discussion about this).

Let $K \subset \mathbb{C}^m$ be compact. Then the Robin constant of K is defined as (see [7])

$$\gamma(K) = \limsup_{z \rightarrow \infty} [V_E^*(z) - \log |z|],$$

and the \mathbb{C}^m -capacity of K is

$$(1.1) \quad C(K) = e^{-\gamma(K)}.$$

Let $K \subset \{z \in \mathbb{C}^m : |z| \leq s\}$ be compact and non-pluripolar, where $s > 0$. Then for any $0 < t < 1$ we have

$$(1.2) \quad \sup_{|z| \leq s} V_K^*(z) \leq m \frac{1+t}{(1-t)^{2m-1}} (\log s + \log \frac{1}{t} - \log C(K)).$$

In fact, the proof of Theorem on page 319 of [12] gives that

$$\int_{S^{2m-1}} V_K^*(rz) d\sigma(z) \leq m(\log r - \log C(K)),$$

for all $r \geq s$, where $d\sigma$ is the normalized area measure on the unit sphere S^{2m-1} in \mathbb{C}^m . Since V_K^* is a non-negative PSH function, if in the above inequality we choose $r = s/t$ and apply Harnack inequality (see for example Chapter 3 in [1]), we obtain

$$\begin{aligned} \sup_{|z| \leq s} V_K^*(z) &\leq \frac{1+t}{(1-t)^{2m-1}} \int_{S^{2m-1}} V_K^*(sz/t) d\sigma(z) \\ &\leq m \frac{1+t}{(1-t)^{2m-1}} (\log s + \log \frac{1}{t} - \log C(K)). \end{aligned}$$

Lemma 1. *Let $E \subset \mathbb{C}^m$ be closed and satisfy the following condition: there exists $0 < \beta \leq 1$ such that*

$$(1.3) \quad \limsup_{R \rightarrow \infty} \frac{\log C(E_R)}{\log R} \geq \beta > 1 - \frac{1}{m},$$

where $C(E_R)$ is determined from formula (1.1).

Let (P_n) be a sequence of polynomials and $k_n \geq \deg(P_n)$. Assume that for all $z \in E$ we have

$$\limsup_{n \rightarrow \infty} |P_n(z)|^{1/k_n} \leq h(|z|)$$

where h is a positive real function such that

$$(1.4) \quad \limsup_{R \rightarrow \infty} \frac{\log h(R)}{\log R} \leq \tau < \infty.$$

For all $\lambda \in S^{2m-1}$, we define

$$\exp\left\{\limsup_{n \rightarrow \infty} \frac{1}{2\pi k_n} \int_0^{2\pi} \log |P_n(e^{i\theta} \lambda)| d\theta\right\} =: C_\lambda.$$

Then for any $w \in \mathbb{C}$ and $\lambda \in S^{2m-1}$ we have

$$\limsup_{n \rightarrow \infty} |P_n(w\lambda)|^{1/k_n} \leq C_\lambda (1 + |w|)^{\tau/[1-m(1-\beta)]}.$$

Corollary 1. *Let $E \subset \mathbb{C}^m$ be closed. Then E is non-thin at ∞ if it satisfies the following condition: There exists $0 < \beta \leq 1$ such that*

$$\limsup_{R \rightarrow \infty} \frac{\log C(E_R)}{\log R} \geq \beta > 1 - \frac{1}{m},$$

where $C(E_R)$ is determined from formula (1.1).

Corollary 2. *Let Δ be a non-empty closed subset of \mathbb{P}^{m-1} and $F = \bigcup_{\lambda \in \Delta} L_\lambda$ where L_λ is the complex line going through 0 with direction λ . Assume that*

$$\limsup_{R \rightarrow \infty} \frac{\log C(F_R)}{\log R} = \gamma > 1 - \frac{1}{m},$$

where as before $F_R = F \cap \{z \in \mathbb{C}^m : |z| \leq R\}$.

Let $E \subset \mathbb{C}^m$ be closed and satisfy the following condition: there exists a constant $0 < \beta \leq 1$ with

$$m(1 - \gamma) < \beta.$$

such that

$$(1.5) \quad \liminf_{s \rightarrow \infty} \frac{1}{\log s} \frac{1}{2\pi} \int_0^{2\pi} V_{E_s}^*(se^{it}\lambda) dt \leq 1 - \beta$$

for all $\lambda \in \Delta$. Then E is non-thin at ∞ .

Interesting questions may arise such as whether we have a Wiener criterion for non-thinness at ∞ in \mathbb{C}^m , or can we extend the above results to sequences of entire functions of bounded orders as done in [3] for one dimensional. An example in [4], giving lower bounds for a constant in another result of Taylor in the cited paper [12], suggests that the best lower bound for β in Lemma 1 and Corollary 1 may be $1 - 1/m$. If the previous claim is true, it means that there are sets in \mathbb{C}^m with "big" capacities yet being thin at ∞ when $m > 1$. We hope to return to these issues in a future paper.

The rest of this paper is devoted to proving the results stated above.

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2. PROOFS OF THE RESULTS

Lemma 2. a) Let E be a subset of \mathbb{C}^m . If E is non-thin at ∞ and A is pluripolar then $E \setminus A$ is non-thin at ∞ .

b) Let $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$ be arbitrary sets. Then $E \times F \subset \mathbb{C}^m \times \mathbb{C}^n$ is non-thin at ∞ iff $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$ are non-thin at ∞ .

Proof. a) Define $F = E \setminus A$. For any $R > 0$ the set $E_R = E \cap \{z \in \mathbb{C}^m : |z| \leq R\}$ is bounded. Hence by Corollary 5.2.5 in [4] we have

$$V_{E_R}^* = V_{F_R}^*.$$

Fix a sequence $R_n \rightarrow \infty$, using the same argument as in the proof of Theorem 2 (see below), there exists a pluripolar set B such that

$$\lim_{n \rightarrow \infty} V_{F_{R_n}}(z) = \lim_{n \rightarrow \infty} V_{F_{R_n}}^*(z) = \lim_{n \rightarrow \infty} V_{E_{R_n}}^*(z) = \lim_{n \rightarrow \infty} V_{E_{R_n}}(z) = 0.$$

for all $z \in \mathbb{C}^m \setminus B$. Since $V_{F_{R_n}}(z)$ are PSH, the previous equality is then also true for $z \in B$, thus proves a).

b) Let $R > 0$. Then one can check easily that

$$\max\{V_{E_R}(z), V_{F_R}(w)\} \leq V_{E_R \times F_R}(z, w) \leq V_{E_R}(z) + V_{F_R}(w).$$

where $z \in \mathbb{C}^m$, $w \in \mathbb{C}^n$. This completes the proof of case b). \square

Now we proceed to proving Theorem 2.

Proof. In this proof fix a sequence $R_n \nearrow \infty$.

(2 \Rightarrow 1) Since E is closed, for any $R > 0$ we have E_R is compact.

Fixed $z_0 \in \mathbb{C}^m$. By Siciak's theorem (see Theorem 5.1.7 in [4]) there exists a sequence of polynomials (P_n) of degree (k_n) such that

$$\|P_n\|_{E_{R_n}}^{1/k_n} \leq 1,$$

for all $n = 1, 2, \dots$, and

$$(2.1) \quad \lim_{n \rightarrow \infty} \log |P_n(z_0)|^{1/k_n} = \lim_{n \rightarrow \infty} V_{E_{R_n}}(z_0).$$

Then for any $z \in E$ we have

$$\limsup_{n \rightarrow \infty} |P_n(z)|^{1/k_n} \leq 1.$$

Hence by assumption that Statement 2 is true, we have

$$\limsup_{n \rightarrow \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in \mathbb{C}^m$. In particular, with $z = z_0$ we get from (2.1) that

$$\lim_{n \rightarrow \infty} V_{E_{R_n}}(z_0) \leq 0.$$

Since $z_0 \in \mathbb{C}^m$ is arbitrary and obviously $V_E(z) \geq 0$ for all z , we get Statement 1.

(1 \Rightarrow 2) We use the ideas in [8]. Assume that Statement 1 is true. Consider any sequence (P_n) of polynomials and $k_n \geq \deg(P_n)$ such that

$$(2.2) \quad \limsup_{n \rightarrow \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in E$. For each n define

$$v_n(z) = \log |P_n(z)|^{1/k_n}.$$

Note that Statement 1 implies that for R large enough then E_R is non-pluripolar. Fix $R > 0$ such that E_R is non-pluripolar.

For any $h, l \in \mathbb{N}$ define

$$E_R^{h,l} = \bigcap_{n=h}^{\infty} \{z \in E_R : v_n(z) < \frac{1}{l}\}.$$

Then $E_R^{h,l} \subset E_R^{h+1,l}$ and from (2.2)

$$(2.3) \quad \bigcup_{h=1}^{\infty} E_R^{h,l} = E_R$$

for any $l \in \mathbb{N}$.

By definition of the pluricomplex Green function, for any $h, l \in \mathbb{N}$, $n \geq h$ and $z \in \mathbb{C}^m$

$$v_n(z) \leq V_{E_R^{h,l}}(z) + \frac{1}{l} \leq V_{E_R^{h,l}}^*(z) + \frac{1}{l}.$$

Hence take the limsup of the above inequality as $n \rightarrow \infty$ we get

$$v(z) = \limsup_{n \rightarrow \infty} v_n(z) \leq V_{E_R^{h,l}}(z) + \frac{1}{l} \leq V_{E_R^{h,l}}^*(z) + \frac{1}{l},$$

for all $h, l \in \mathbb{N}$, $z \in \mathbb{C}^m$. Take the limit of this inequality as $h \rightarrow \infty$, using (2.2), we see from Corollary 5.2.6 in [4] that

$$v(z) \leq \lim_{h \rightarrow \infty} V_{E_R^{h,l}}^*(z) + \frac{1}{l} = V_{E_R}^*(z) + \frac{1}{l},$$

for all $l \in \mathbb{N}$, $z \in \mathbb{C}^m$, $R > 0$. Since l is arbitrary, we get

$$(2.4) \quad v(z) \leq V_{E_R}^*(z),$$

for all $z \in \mathbb{C}^m$, $R > 0$.

For each $n \in \mathbb{N}$ define

$$A_n = \{z \in \mathbb{C}^m : V_{E_{R_n}}(z) < V_{E_{R_n}}^*(z)\}$$

then A_n is pluripolar thus has Lebesgue measure zero, and $V_{E_{R_n}}(z) = V_{E_{R_n}}^*(z)$ for $z \in \mathbb{C}^m \setminus A_n$. Hence

$$A = \bigcup_{n=1}^{\infty} A_n$$

is also pluripolar and of Lebesgue measure zero, and we have $V_{E_{R_n}}(z) = V_{E_{R_n}}^*(z)$ for $z \in \mathbb{C}^m \setminus A$ and $n \in \mathbb{N}$. Hence for $z \in \mathbb{C}^m \setminus A$, apply (2.4) we get

$$v(z) \leq \lim_{n \rightarrow \infty} V_{E_{R_n}}^*(z) = \lim_{n \rightarrow \infty} V_{E_{R_n}}(z) = 0$$

for all $z \in \mathbb{C}^m \setminus A$. Since A is of Lebesgue measure zero, by definition of $v(z)$, we see that $v(z) \leq 0$ for all $z \in \mathbb{C}^m$. This completes the proof of Theorem 2. \square

Now we prove Theorem 3.

Proof. (\Rightarrow) Let E be a closed subset of \mathbb{C}^m . Assume that $V_E \equiv 0$. We will show that any open neighborhood of E is non-thin at ∞ . Let F be any open neighborhood of E . Then for any $R > 0$ since E_R is compact, F is open and contains E_R , we can find $1 > \epsilon = \epsilon_R > 0$ such that

$$E_{R,\epsilon} = \{z \in \mathbb{C}^m : \text{dist}(z, E_R) \leq \epsilon\} \subset F.$$

By Corollary 5.1.5 in [4] we have $V_{E_{R,\epsilon}}^*(z) = 0$ for $z \in E_{R,\epsilon} \supset E_R$. Now it is obvious that $E_{R,\epsilon} \subset F_{R+1}$ hence for $z \in E_R$

$$(2.5) \quad V_{F_{R+1}}^*(z) = 0.$$

Define

$$v(z) = \lim_{R \rightarrow \infty} V_{F_R}^*(z),$$

then v is the limit of a decreasing sequence of PSH functions hence v is itself PSH. By (2.5) for $z \in E$ we have $v(z) \equiv 0$. Thus by definition of the pluricomplex Green function

$$v(z) \leq V_E(z) = 0$$

for all $z \in \mathbb{C}^m$. This shows that F is non-thin at ∞ .

(\Leftarrow) Assume that E is an arbitrary subset of \mathbb{C}^m with $V_E \not\equiv 0$. Then $V_E(z_0) > 0$ for some $z_0 \in \mathbb{C}^m$, hence by definition of the pluricomplex Green function, there exist a function $u \in \mathbb{L}$ such that $u(z) \leq 0$ for $z \in E$, and $u(z_0) > 0$. Define

$$F = \{z \in \mathbb{C}^m : u(z) < u(z_0)/2\}.$$

F is open because u is upper-semicontinuous, and $E \subset F$ because $u|_E \leq 0$ and $u(z_0) > 0$. Now $u(z) < u(z_0)/2$ for $z \in F$, hence we have

$$u(z) \leq V_F(z) + u(z_0)/2$$

for all $z \in \mathbb{C}^m$. In particular choose $z = z_0$ we have $V_F(z_0) \geq u(z_0)/2 > 0$, hence F is thin at ∞ . \square

We conclude this section presenting the proofs of Lemma 1 and Corollaries 1 and 2.

Proof. (Of Lemma 1) Define

$$u(z) = \limsup_{n \rightarrow \infty} \log |P_n(z)|^{1/k_n},$$

and let $u^*(z)$ be its upper-semicontinuous regularization. Then $u^*(z) \in \mathbb{L}$.

Now we define

$$(2.6) \quad \kappa_0 = \limsup_{z \in \mathbb{C}^m, z \rightarrow \infty} \frac{u^*(z)}{\log |z|}.$$

For any $R > 0$, by the definition of the pluricomplex Green function

$$(2.7) \quad u^*(z) \leq \kappa_0 V_{E_R}(z) + \log h(R).$$

For any $\lambda \in S^{2m-1}$ define

$$\kappa(\lambda) = \limsup_{w \in \mathbb{C}, w \rightarrow \infty} \frac{u(w\lambda)}{\log |w|} \leq \kappa_0.$$

Fixed $\lambda \in S^{m-1}$. By definition

$$u(w\lambda) = \limsup_{n \rightarrow \infty} \frac{1}{k_n} \log |P_n(w\lambda)|$$

for all $w \in \mathbb{C}$. Hence using (2.7) and (1.2), for any $s > 0$, $0 < t < 1$ we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{k_n} \log |P_n(se^{i\theta}\lambda)| d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} u(se^{i\theta}\lambda) d\theta \\ &\leq \kappa_0 \frac{1}{2\pi} \int_0^{2\pi} V_{E_s}(se^{i\theta}\lambda) d\theta + \log h(s) \\ &\leq \kappa_0 m \frac{1+t}{(1-t)^{2m-1}} (\log s + \log \frac{1}{t} - \log C(E_s)) + \log h(s). \end{aligned}$$

By the previous inequality and (1.4), we get for any $0 < t < 1$:

$$(2.8) \quad \liminf_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{\log s} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{k_n} \log |P_n(se^{i\theta}\lambda)| d\theta \leq \kappa_0 m \frac{1+t}{(1-t)^{2m-1}} (1-\beta) + \tau.$$

Take the limit $t \rightarrow 0^+$ in the previous inequality we obtain

$$(2.9) \quad \liminf_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{\log s} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{k_n} \log |P_n(se^{i\theta}\lambda)| d\theta \leq \kappa_0 m (1-\beta) + \tau.$$

Then by Lemma 2 in [3] we have

$$(2.10) \quad \kappa(\lambda) \leq \kappa_0 m (1-\beta) + \tau.$$

Since (2.10) is true for any $\lambda \in S^{2m-1}$, by definition (2.6) and Lemma 2 in [3] we have

$$\kappa_0 \leq \kappa_0 m (1-\beta) + \tau,$$

or equivalently

$$\kappa_0 \leq \frac{\tau}{1-m(1-\beta)}.$$

From the above inequality, use Lemma 2 in [3] we get the conclusion of Lemma 1. \square

Proof. (Of Corollary 1) Let (P_n) be a sequence of polynomials and let $k_n \geq \deg(P_n)$ satisfying

$$u(z) = \limsup_{n \rightarrow \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in E$. Apply Lemma 1 for the set E , the sequences (P_n) and (k_n) , and $\tau = 0$, we get

$$\limsup_{n \rightarrow \infty} |P_n(z)|^{1/k_n} \leq 1,$$

for all $z \in \mathbb{C}^m$. Then Theorem 2 implies that E is non-thin at ∞ . \square

Proof. (Of Corollary 2)

From the assumptions, we have that E_R is non-pluripolar for $R > 0$ large enough.

Let (P_n) be a sequence of polynomials and let $k_n \geq \deg(P_n)$ satisfying

$$u(z) = \limsup_{n \rightarrow \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in E$. Define

$$\kappa_0 = \limsup_{z \in \mathbb{C}^m, |z| \rightarrow \infty} \frac{u^*(z)}{\log |z|} \leq 1,$$

and

$$\kappa_1 = \sup_{\lambda \in \Delta} \limsup_{w \in \mathbb{C}, |w| \rightarrow \infty} \frac{u^*(w\lambda)}{\log |w|} \leq \kappa_0.$$

Apply Lemma 1 for the set F , the sequences (P_n) and (k_n) , and $\tau = \kappa_1$ we obtain

$$\kappa_0 \leq \frac{\kappa_1}{1 - m(1 - \gamma)}.$$

Using assumption (1.5), apply formula (2.10) for all $\lambda \in \Delta$, for the set E , the sequences (P_n) and (k_n) , and $\tau = 0$, we obtain

$$\kappa_1 \leq \kappa_0(1 - \beta).$$

Hence we have

$$\frac{\kappa_1}{1 - \beta} \leq \kappa_0 \leq \frac{\kappa_1}{1 - m(1 - \gamma)}.$$

From our assumption that $\beta > m(1 - \gamma)$, the above inequality implies that $\kappa_0 = \kappa_1 = 0$. Hence $u^*(z)$ is a constant. Moreover, $u^*(z) \leq 0$ on the set E which is non-pluripolar hence $u^*(z) \leq 0$ for all $z \in \mathbb{C}^m$. By Theorem 2, E is non-thin at ∞ . \square

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