TIKHONOV'S REGULARIZATION TO DECONVOLUTION PROBLEM*

Dang Duc Trong[a], Cao Xuan Phuong[b], Truong Trung Tuyen[c], and Dinh Ngoc Thanh[a]

[a] Faculty of Mathematics and Computer Science, Ho Chi Minh City National University, 227 Nguyen Van Cu, District 5, Ho Chi Minh City, Viet Nam.

[b] Faculty of Mathematics and Statistics, Ton Duc Thang University, Nguyen Huu Tho, District 7, Ho Chi Minh City, Viet Nam.

[c] Department of Mathematics, Indiana University, Bloomington, IN 47405, USA.

Abstract

We are interested in estimating the pdf f of i.i.d. random variables $X_1, \ldots, X_n$ from the model $Y_j = X_j + Z_j$, where Z_j are unobserved error random variables, distributed with the density function g and independent of X_j. This problem is known as the deconvolution problem in nonparametric statistics. The most popular method of solving the problem is the kernel one in which, we assume $g^{\mathrm{ft}}(t) \neq 0$, for all $t \in \mathbb{R}$, where $g^{\mathrm{ft}}(t)$ is the Fourier transform of g. The more general case in which $g^{\mathrm{ft}}(t)$ may have real zeros has not been considered much. In this paper, we shall consider this case. By estimating Lebesgue measure of the low level sets of g^{ft} and combining with the Tikhonov's regularization method, we give an approximation f_n to the density function f and evaluate the rate of convergence of $\sup_{g \in \mathcal{G}_{s_0,\gamma,M,T}} \sup_{f \in \mathcal{F}_{q,K}} \mathbb{E} \|f_n - f\|^2_{L^2(\mathbb{R})}$. A lower bound for this quantity is also provided.

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1 Introduction

In this paper, we are interested in estimating the density function f of the random variables i.i.d $X_1, X_2, \ldots, X_n$ based on the direct random variables $Y_1, Y_2, \ldots, Y_n$ from model

$$Y_j = X_j + Z_j, \quad j = 1, 2, \ldots, n. \tag{1}$$

Here Z_j are unobserved error random variables, distributed with the density function g and independent of X_j. We know that if h is the probability density function of Y_j, then we have the relation

$$h = f * g \tag{2}$$

where the symbol $*$ denotes the convolution of two functions f and g,

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-t)\, g(t)\, dt, \quad x \in \mathbb{R}.$$

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We denote the Fourier transform of the function f by

$$f^{\text{ft}}(t) = \int_{-\infty}^{+\infty} f(x) e^{itx} dx, \quad t \in \mathbb{R}. \tag{3}$$

Put

$$NZ_g = \left\{ t \in \mathbb{R} : g^{\text{ft}}(t) \neq 0 \right\}.$$

Informally, if h is known, we can apply the Fourier transform to both sides of (2) to get

$$f^{\text{ft}} = \frac{h^{\text{ft}}}{g^{\text{ft}}} \quad \text{for all} \quad t \in NZ_g. \tag{4}$$

Then using the inverse Fourier transform, we can find f. This is a classical problem in Analysis.

In practical situations, we do not have the density function h. We only have the observations Y_j, $j = 1, \ldots, n$. The problem of recovering f from observations Y_j is called the *deconvolution problem in statistics* or *deconvolution problem* for short. Equation (2) is an integral equation and solving (2) is a typically ill-posed problem.

A specific deconvolution problem is the one of consistency. To prove a deconvolution problem is consistent, we have to show that there exists a sequence of estimators $\{f_n\}$ such that

$$\lim_{n \to +\infty} \mathbb{E} \| f_n(\cdot; Y_1, \ldots, Y_n) - f \|_X = 0$$

where X is an appropriate Banach space.

In fact, the simplest case is $NZ_g = \mathbb{R}$. In this case, there are many methods to construct the estimator $f_n(x; Y_1, \ldots, Y_n)$. Kernel estimation is one of the most popular approach to deal with the deconvolution problem. In this method, one estimates the density function f by the estimator

$$f_n(x; Y_1, \ldots, Y_n) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \frac{K^{\text{ft}}(tb)}{g^{\text{ft}}(t)} \frac{1}{n} \sum_{j=1}^{n} e^{itY_j} dt, \tag{5}$$

where K is a kernel function and K^{ft} is compactly supported. This method was first introduced in the papers of Carroll and Hall [9], Stefanski and Carroll [11], Fan [12], [13]. The estimator (5) has known as the standard deconvolution kernel density. We note that the estimator (5) is defined as $g^{\text{ft}}(t) \neq 0$ for all $t \in \mathbb{R}$, and so the condition $NZ_g = \mathbb{R}$ has become common in deconvolution topics. In fact, the density functions g often satisfy

$$\left| g^{\text{ft}}(t) \right| \geq C \left(1 + t^2\right)^{-\alpha} \exp\left\{ -C_0 |t|^{\gamma} \right\}$$

where C, $C_0 > 0$, $\alpha \geq 0$, $\gamma \geq 0$ and $\alpha + \gamma > 0$. Similarly, in case of bivariate random variables, Goldenshluger [1] also assumed that

$$\min_{|t| \leq v} \left| g^{\text{ft}}(t) \right| \geq C \exp\left\{ -C_0 v^2 \right\}, \quad \forall v > 0.$$

However, there are many important density functions which do not satisfy $NZ_g = \mathbb{R}$, e.g., the uniform densities, the self-convolved uniform densities or the convolution of an arbitrary density function with a uniform density.

Deconvolution problem in the case $NZ_g \neq \mathbb{R}$ is very difficult. According to our knowledge, there is only a few articles mentioning this case. The first paper which considered this problem is Devroye [10]. The consistency was established with respect to the $L^1(\mathbb{R})$-norm. Using the technique of truncation, he constructed a consistent estimator f_n for the target density f when the Fourier transform g^{ft} vanishes on a Lebesgue-zero-set,

$$\begin{aligned} f_n(x; Y_1, \ldots, Y_n) &= \frac{1}{2\pi} \operatorname{Re}\left\{ \int_{\mathbb{R}\setminus A_r} e^{-itx} \frac{K(th)}{g^{\text{ft}}(t)} \frac{1}{n} \sum_{j=1}^{n} e^{itY_j} dt \right\}, \quad |x| < T, \\ &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad |x| \geq T, \end{aligned}$$

where $A_r = \left\{ t \in \mathbb{R} : \left| g^{\text{ft}}(t) \right| < r \right\}$, $r > 0$, $h > 0$ and K^{ft} is compactly supported. However, no convergence rate is provided in Devroye [10].

In Meister [3], the density deconvolution is also considered in case the target density f is contained in the class of densities which satisfy $\int_{-S}^{S} f(x)\, dx = 1$ and

$$\int_{-\infty}^{+\infty} \left| f^{\text{ft}}(t) \right|^2 \left(1 + t^2\right)^{\beta} dt \leq C$$

with S, C, $\beta > 0$ whereas the error density belongs to the class of densities which has $\left| g^{\text{ft}}(t) \right| \geq \mu$ for $t \in [-\nu, \nu]$ and $\|g\|_{\infty} \leq C$. The rate of uniform convergence of MISE(f_n, f) is $O\left((\ln n)^{-2\beta(1-\delta)} (\ln \ln n)^{2\beta} \right)$ with $\delta \in [0, 1)$. This rate is only derived when the sample size n is chosen such that $S \in \left[\frac{O\left((\ln n)^{\delta}\right)}{O(1)}; O\left((\ln n)^{\delta} \right) \right]$. Actually, this condition is difficult to verify because S is not known exactly and so we cannot choose n exactly in general.

In Groeneboom and Jongbloed [14], the authors focused on considering the deconvolution problem in a uniform density model. By choosing a suitable bandwidth, they proved that it is possible to construct a kernel estimator of target density f if f has a finite left endpoint. In Hall and Meister [15], the authors have given an approach to solve the deconvolution problem in case $NZ_g \neq \mathbb{R}$. To avoid division by zero, the authors replaced $g^{\text{ft}}(t)$ by the maximum of $g^{\text{ft}}(t)$ and $h_n(t) = n^{-\xi}|t|^{\rho}$ with $\xi > 0$, $\rho > 0$. The function $h_n(t)$ as above is called the "ridge function". An estimator for the density f is defined by

$$f_n(x) = \operatorname{Re}\left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \frac{g^{\text{ft}}(-t) \left| g^{\text{ft}}(t) \right|^r}{\left(\max\left\{ \left| g^{\text{ft}}(t) \right| ; h_n(t) \right\} \right)^{r+2}} \frac{1}{n} \sum_{j=1}^{n} e^{itY_j} dt \right\}, \tag{6}$$

with $r \geq 0$. The optimal rates of estimation are provided. Recently, using a modified kernel method, Delaigle and Meister [5] also gave a similar result. We see that the condition imposed on g^{ft} is very strict. In these papers, the density function g is assumed to satisfy

$$\left|g^{\text{ft}}(t)\right| \geq c|\sin(kt)|^{\nu}(1+|t|)^{-\alpha}\exp\left\{-d|t|^{\beta}\right\}, \quad t \in \mathbb{R}, \tag{7}$$

with $k > 0$, $c > 0$, $d > 0$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta > 0$, $\nu > 0$. In this case, $\mathbb{R} \backslash NZ_g \subset \left\{\frac{n\pi}{k} : n \in \mathbb{Z}\right\}$. In other words, the positions of zeros of g^{ft} are fixed. If g is a uniform density then (7) holds, but if g is an arbitrary density function then the condition (7) often does not satisfy.

Motivated by this problem, in the present paper, we shall consider the deconvolution problem in case the Fourier transform of error distribution has zeros on the real line, with no specific constraints on NZ_g. Applying Tikhonov's regularization, we introduce an estimation procedure for the target density function. Using properties of entire functions and some results from harmonic analysis, we consider the low level sets of the function g and give the convergence rate of our procedure.

The rest of our paper consists of three sections. In Section 2, we shall present the Tikhonov's regularization; and use it to give a result of consistency and an estimation for probability density functions. In Section 3, we state and prove approximation results and provide a lower bound for error of the estimator. In Section 4, by estimating of Lebesgue measure of the low level sets of the Fourier transform of g, we prove Lemma 3.1 which is stated in Section 3.

2 Tikhonov's regularization

As discussed in Section 1, we know that Problem (2) is typically ill-posed and a regularization is required. In the theory of ill-posed problems, a method of regularization which is often used for the deconvolution problem is the Tikhonov's regularization. In this method, we shall approximate f^{ft} by a function having the form φg^{ft} where φ is often called the"filter function". In fact, we consider the linear operator $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $A(\varphi) = \varphi g^{\text{ft}}$ for all $\varphi \in L^2(\mathbb{R})$. For each $\delta > 0$, we consider the *Tikhonov's functional*

$$J_\delta(\varphi) = \left\|A\varphi - h^{\text{ft}}\right\|^2_{L^2(\mathbb{R})} + \delta\|\varphi\|^2_{L^2(\mathbb{R})}, \ \varphi \in L^2(\mathbb{R}). \tag{8}$$

We shall find the function φ minimizing J_δ. As known, J_δ attains its minimum at a unique minimum function $\varphi_\delta \in L^2(\mathbb{R})$. This minimum φ_δ is the unique solution of the equation

$$\delta\varphi_\delta + (A^*A)(\varphi_\delta) = A^*\left(h^{\text{ft}}\right), \tag{9}$$

where $A^*: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the adjoint operator of A (see Theorem 2.11, Section 2.2 in [6]). From (9), we get

$$\delta\varphi_\delta + \left|g^{\rm ft}\right|^2 \varphi_\delta = \overline{g^{\rm ft}}\, h^{\rm ft}$$

and so we have the approximation

$$\varphi_\delta = \frac{\overline{g^{\rm ft}}}{\delta + \left|g^{\rm ft}\right|^2}$$

to the Fourier transform $f^{\rm ft}$ of the density function f. After that, using the inverse Fourier transform gives

$$f_\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx} \frac{\overline{g^{\rm ft}(t)}\, h^{\rm ft}(t)}{\delta + \left|g^{\rm ft}(t)\right|^2} dt, \quad x \in \mathbb{R}. \tag{10}$$

This can be seen as an estimator for the density function f.

As mentioned, in practical situations we do not have the density function h, we have only the observations $Y_1, \ldots, Y_n$. Thus, we cannot use directly the formula (10) to make an approximation for f. However, in case we have the i.i.d. observations $Y_1, \ldots, Y_n$, since

$$\mathbb{E}\left(\frac{1}{n}\sum_{j=1}^{n} e^{itY_j}\right) = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left(e^{itY_j}\right) = h^{\rm ft}(t),$$

we can replace $h^{\rm ft}(t)$ in (10) by the quantity

$$\Psi(t; Y_1, ..., Y_n) = \frac{1}{n}\sum_{j=1}^{n} e^{itY_j}. \tag{11}$$

It suggests an approximation for the density function f based on $Y_1, \ldots, Y_n$ as follows

$$L_{\delta,g}(x; Y_1, \ldots, Y_n) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx} \frac{\overline{g^{\rm ft}(t)}}{\delta + \left|g^{\rm ft}(t)\right|^2} \frac{1}{n}\sum_{j=1}^{n} e^{itY_j} dt. \tag{12}$$

We have the following general estimate for error $\mathbb{E}\left\|L_{\delta,g} - f\right\|_{L^2(\mathbb{R})}^2$.

Lemma 2.1 *Let $\delta > 0$, $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be the density function of error random variables and $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be the solution of Problem (2). Then*

$$\begin{aligned}\mathbb{E}\left\|L_{\delta,g} - f\right\|_{L^2(\mathbb{R})}^2 = {} & \frac{1}{2\pi n}\int_{-\infty}^{+\infty} \frac{\left|g^{\rm ft}(t)\right|^2}{\left(\delta + \left|g^{\rm ft}(t)\right|^2\right)^2}\left(1 - \left|f^{\rm ft}(t)\, g^{\rm ft}(t)\right|^2\right) dt \\ & + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|f^{\rm ft}(t)\right|^2 \left(\frac{\delta}{\delta + \left|g^{\rm ft}(t)\right|^2}\right)^2 dt.\end{aligned} \tag{13}$$

Proof. From (12), we get

$$L_{\delta,g}^{\text{ft}}(t) = \frac{\overline{g^{\text{ft}}(t)}}{\delta + \left|g^{\text{ft}}(t)\right|^2} \frac{1}{n} \sum_{j=1}^{n} e^{itY_j}, \quad t \in \mathbb{R}.$$

Applying the Parseval's identity, the Fubini's theorem and the equality

$$\mathbb{E}\left|L_{\delta,g}^{\text{ft}}(t) - f^{\text{ft}}(t)\right|^2 = \operatorname{Var} L_{\delta,g}^{\text{ft}}(t) + \left|\mathbb{E}L_{\delta,g}^{\text{ft}}(t) - f^{\text{ft}}(t)\right|^2,$$

we derive

$$\begin{aligned}
\mathbb{E}\left\|L_{\delta,g} - f\right\|_{L^2(\mathbb{R})}^2 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbb{E}\left|L_{\delta,g}^{\text{ft}}(t) - f^{\text{ft}}(t)\right|^2 dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \operatorname{Var} L_{\delta,g}^{\text{ft}}(t)\, dt + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|\mathbb{E}L_{\delta,g}^{\text{ft}}(t) - f^{\text{ft}}(t)\right|^2 dt.
\end{aligned}$$

As the $Y_1, Y_2, \ldots, Y_n$ are i.i.d. random variables, we get

$$\begin{aligned}
\int_{-\infty}^{+\infty} \operatorname{Var} L_{\delta,g}^{\text{ft}}(t) dt &= \frac{1}{n^2} \int_{-\infty}^{+\infty} \left|\frac{\overline{g^{\text{ft}}(t)}}{\delta + \left|g^{\text{ft}}(t)\right|^2}\right|^2 \sum_{j=1}^{n} \operatorname{Var}\left(e^{itY_j}\right) dt \\
&= \frac{1}{n} \int_{-\infty}^{+\infty} \left|\frac{\overline{g^{\text{ft}}(t)}}{\delta + \left|g^{\text{ft}}(t)\right|^2}\right|^2 \operatorname{Var}\left(e^{itY_1}\right) dt \\
&= \frac{1}{n} \int_{-\infty}^{+\infty} \frac{\left|g^{\text{ft}}(t)\right|^2}{\left(\delta + \left|g^{\text{ft}}(t)\right|^2\right)^2} \left(\mathbb{E}\left|e^{itY_1}\right|^2 - \left|\mathbb{E}\left(e^{itY_1}\right)\right|^2\right) dt \\
&= \frac{1}{n} \int_{-\infty}^{+\infty} \frac{\left|g^{\text{ft}}(t)\right|^2}{\left(\delta + \left|g^{\text{ft}}(t)\right|^2\right)^2} \left(1 - \left|h^{\text{ft}}(t)\right|^2\right) dt \\
&= \frac{1}{n} \int_{-\infty}^{+\infty} \frac{\left|g^{\text{ft}}(t)\right|^2}{\left(\delta + \left|g^{\text{ft}}(t)\right|^2\right)^2} \left(1 - \left|f^{\text{ft}}(t)\, g^{\text{ft}}(t)\right|^2\right) dt
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\infty}^{+\infty} \left|\mathbb{E}L_{\delta,g}^{\text{ft}}(t) - f^{\text{ft}}(t)\right|^2 dt &= \int_{-\infty}^{+\infty} \left|\frac{\overline{g^{\text{ft}}(t)}}{\delta + \left|g^{\text{ft}}(t)\right|^2} \mathbb{E}\left(e^{itY_1}\right) - f^{\text{ft}}(t)\right|^2 dt \\
&= \int_{-\infty}^{+\infty} \left|\frac{\overline{g^{\text{ft}}(t)}}{\delta + \left|g^{\text{ft}}(t)\right|^2} h^{\text{ft}}(t) - f^{\text{ft}}(t)\right|^2 dt \\
&= \int_{-\infty}^{+\infty} \left|\frac{\left|g^{\text{ft}}(t)\right|^2}{\delta + \left|g^{\text{ft}}(t)\right|^2} f^{\text{ft}}(t) - f^{\text{ft}}(t)\right|^2 dt \\
&= \int_{-\infty}^{+\infty} \left|f^{\text{ft}}(t)\right|^2 \left(\frac{\delta}{\delta + \left|g^{\text{ft}}(t)\right|^2}\right)^2 dt.
\end{aligned}$$

Combining the above equalities, we get the conclusion of Lemma. ■

The consistency of the problem with respect to the $L^1(\mathbb{R})$-norm was studied in Devroye [10]. Meister [4] also gave a very general consistency result in $L^2(\mathbb{R})$-weighted norm (NZ_g is assumed to be dense in $\mathbb{R}$). Now, we shall give a consistency result in $L^2(\mathbb{R})$-norm with a simple estimator and an easy proof.

Theorem 2.2 *Let $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be the density function of the error random variables and $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be the solution of Problem (2). Assume that $m(\mathbb{R}\backslash NZ_g) = 0$ where $m(\cdot)$ is the Lebesgue measure on $\mathbb{R}$. Let (δ_n) be a positive sequence such that $\delta_n \to 0$, $n\delta_n^2 \to +\infty$ as $n \to +\infty$. Then*

$$\lim_{n\to+\infty} \mathbb{E}\left\|L_{\delta_n,g} - f\right\|_{L^2(\mathbb{R})}^2 = 0.$$

Proof. Applying the result (13) of Lemma 2.1, we have

$$\begin{aligned}
\mathbb{E}\left\|L_{\delta_n,g} - f\right\|_{L^2(\mathbb{R})}^2 &= \frac{1}{2\pi n}\int_{-\infty}^{+\infty} \frac{\left|g^{\mathrm{ft}}(t)\right|^2}{\left(\delta_n + \left|g^{\mathrm{ft}}(t)\right|^2\right)^2}\left(1 - \left|f^{\mathrm{ft}}(t)\, g^{\mathrm{ft}}(t)\right|^2\right) dt \\
&+ \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|f^{\mathrm{ft}}(t)\right|^2 \left(\frac{\delta_n}{\delta_n + \left|g^{\mathrm{ft}}(t)\right|^2}\right)^2 dt \\
&\le \frac{1}{2\pi n\delta_n^2}\int_{-\infty}^{+\infty} \left|g^{\mathrm{ft}}(t)\right|^2 dt + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|f^{\mathrm{ft}}(t)\right|^2 \left(\frac{\delta_n}{\delta_n + \left|g^{\mathrm{ft}}(t)\right|^2}\right)^2 dt \\
&= \frac{1}{n\delta_n^2}\|g\|_{L^2(\mathbb{R})}^2 + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|f^{\mathrm{ft}}(t)\right|^2 \left(\frac{\delta_n}{\delta_n + \left|g^{\mathrm{ft}}(t)\right|^2}\right)^2 dt.
\end{aligned}$$

Using the Lebesgue's dominated convergence theorem, we get $\mathbb{E}\left\|L_{\delta_n,g} - f\right\|_{L^2(\mathbb{R})}^2 \to 0$ as $n \to +\infty$. The proof of the theorem is completed. ■

The following theorem will be used in Section 2 to get the main result of our paper.

Theorem 2.3 *Let $\rho > 0$, $\delta > 0$, $R > 0$, let $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be the density function of error random variables and $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be the solution of Problem (2). Then*

$$\mathbb{E}\left\|L_{\delta,g} - f\right\|_{L^2(\mathbb{R})}^2 \le C_1\left(m\left(B_{\rho,R}\right) + \int_{|t|>R}\left|f^{\mathrm{ft}}(t)\right|^2 dt + \frac{\delta^2}{\rho^4} + \frac{1}{n\delta^2}\right), \tag{14}$$

where

$$\begin{aligned}
C_1 &= \frac{1}{2\pi}\max\left\{1; \left\|f^{\mathrm{ft}}\right\|_{L^2(\mathbb{R})}^2; \left\|g^{\mathrm{ft}}\right\|_{L^2(\mathbb{R})}^2\right\}, \\
B_{\rho,R} &= \left\{t \in \mathbb{R} : \left|g^{\mathrm{ft}}(t)\right| < \rho,\ |t| < R\right\}.
\end{aligned}$$

Proof. From (13), we get the estimate

$$\mathbb{E}\left\|L_{\delta,f} - f\right\|_{L^2(\mathbb{R})}^2 \le \frac{1}{2\pi n\delta^2}\int_{-\infty}^{+\infty}\left|g^{\mathrm{ft}}(t)\right|^2 dt + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|f^{\mathrm{ft}}(t)\right|^2\left(\frac{\delta}{\delta + \left|g^{\mathrm{ft}}(t)\right|^2}\right)^2 dt.$$

Now we write

$$\int_{-\infty}^{+\infty} \left|f^{\text{ft}}(t)\right|^2 \left(\frac{\delta}{\delta + \left|g^{\text{ft}}(t)\right|^2}\right)^2 dt = \int_{|t|<R,\, |g^{\text{ft}}(t)|<\rho} \left|f^{\text{ft}}(t)\right|^2 \left(\frac{\delta}{\delta + \left|g^{\text{ft}}(t)\right|^2}\right)^2 dt$$
$$+ \int_{|t|>R,\, |g^{\text{ft}}(t)|<\rho} \left|f^{\text{ft}}(t)\right|^2 \left(\frac{\delta}{\delta + \left|g^{\text{ft}}(t)\right|^2}\right)^2 dt$$
$$+ \int_{|g^{\text{ft}}(t)|>\rho} \left|f^{\text{ft}}(t)\right|^2 \left(\frac{\delta}{\delta + \left|g^{\text{ft}}(t)\right|^2}\right)^2 dt$$

$$\leq \int_{|t|<R,\, |g^{\text{ft}}(t)|<\rho} dt + \int_{|t|>R,\, |g^{\text{ft}}(t)|<\rho} \left|f^{\text{ft}}(t)\right|^2 dt + \left(\frac{\delta}{\rho^2}\right)^2 \int_{|g^{\text{ft}}(t)|>\rho} \left|f^{\text{ft}}(t)\right|^2 dt$$

$$\leq m\left(B_{\rho,R}\right) + \int_{|t|>R} \left|f^{\text{ft}}(t)\right|^2 dt + \frac{\delta^2}{\rho^4} \left\|f^{\text{ft}}\right\|_{L^2(\mathbb{R})}^2 .$$

Therefore,

$$\mathbb{E}\left\|L_{\delta,g} - f\right\|_{L^2(\mathbb{R})}^2 \leq C_1 \left(m\left(B_{\rho,R}\right) + \int_{|t|>R} \left|f^{\text{ft}}(t)\right|^2 dt + \frac{\delta^2}{\rho^4} + \frac{1}{n\delta^2}\right),$$

where $C_1 = \frac{1}{2\pi} \max\left\{1; \left\|f^{\text{ft}}\right\|_{L^2(\mathbb{R})}^2; \left\|g^{\text{ft}}\right\|_{L^2(\mathbb{R})}^2\right\}$. The proof of the theorem is completed. ■

3 Approximation results

The difficulty of applying directly the formula (4) arises in two aspects: one, in reality we cannot have the density function h, and two, even we have h, we cannot compute efficiently the inverse Fourier transform of f^{ft} if the function g^{ft} has zeros on the real axis. Hence, a regularization is in order.

For each entire function ψ, *low level sets* of ψ are defined by $\{z \in \mathbb{C} : |\psi(z)| < \varepsilon\}$, $\varepsilon > 0$. In case g is compactly supported, the set of zeros $g^{\text{ft}}(t)$ affects heavily the recovering of the function f from its Fourier transform. For actually computing the solution f and for the regularization of equation (2), we must know more about the Lebesgue measure of the low level sets of g^{ft}. The latter goes back to the well-known theorem of Cartan about the size of the low level sets $A_\varepsilon = \{z \in \mathbb{C} : |P(z)| < \varepsilon\}$, $\varepsilon > 0$, where $P(z)$ is a polynomial. He proved that A_ε is contained in a finite set of disks whose sum of radius is less than $C\varepsilon^{\frac{1}{n}}$, where n is the degree of $P(z)$ and C is a constant that depends only on the leading coefficient of $P(z)$ and n (see Theorem 3 of §11.2 in [7]). In particular, we have

$$\lim_{\varepsilon \to 0} m\left(\{z \in \mathbb{C} : |P(z)| < \varepsilon\}\right) = 0$$

where $m(\cdot)$ is the Lebesgue measure on $\mathbb{C}$.

We shall use the Cartan's theorem to give an asymptotic estimate on the low level sets $\left\{t \in \mathbb{R} : \left|g^{\mathrm{ft}}(t)\right| < \varepsilon,\ |t| < R_\varepsilon\right\}$ of the function g, and shall apply this estimate to the Tikhonov's regularization of Problem (2).

To get an explicit estimate for $\mathbb{E}\left\|L_{\delta,g} - f\right\|^2_{L^2(\mathbb{R})}$, some prior information to f and g must be assumed. From now on, we assume that f is contained in the set

$$\mathcal{F}_{q,K} = \left\{f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) : f \geq 0,\ \int_{-\infty}^{+\infty} f(x)\,dx = 1,\ \left|f^{\mathrm{ft}}(t)\right|^2 \leq \frac{K}{(1+t^2)^q}\right\}, \quad (15)$$

where $K \geq 1$ and $q > \frac{1}{2}$. The condition $\left|f^{\mathrm{ft}}(t)\right|^2 \leq \frac{K}{(1+t^2)^q}$ imposed on the density $f \in \mathcal{F}_{q,K}$ as in (15) is quite natural. It is equivalent to the condition that the density function f is in the Sobolev space $W^{q,1}(\mathbb{R})$.

Moreover, let $s_0 > 0$, $\gamma \in (1,2)$, $M \geq 1$ and $T > \frac{1}{2}$, we assume that the density function g of the error random variables belongs to the class of functions

$$\mathcal{G}_{s_0,\gamma,M,T} = \left\{g \in L^2(\mathbb{R}) : g \geq 0,\ \int_{-\infty}^{+\infty} g(x)\,dx = 1,\ \int_{-\infty}^{+\infty} g(t)\, e^{s_0|t|^\gamma}dt \leq M, \|g\|^2_{L^2(\mathbb{R})} \leq T\right\}. \quad (16)$$

We note that the density function $g(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ of Gauss distribution and compactly supported density functions are in $\mathcal{G}_{s_0,\gamma,M,T}$. Not all of such functions satisfies $NZ_g = \mathbb{R}$.

For each $\varepsilon > 0$, we put

$$s_\varepsilon = \inf\left\{s > 0 : \int_{|t|\geq s} |g(t)|\,dt \leq \varepsilon\right\}. \quad (17)$$

Lemma 3.1 *Let $s_0 > 0$, $\lambda \in (1,2)$, $M \geq 1$, $T > \frac{1}{2}$, $\beta \in (0,1)$, $q > \frac{1}{2}$ and let $g \in \mathcal{G}_{s_0,\gamma,M,T}$ be the density function of error random variables. For $\varepsilon > 0$ small enough, choose R_ε to satisfy*

$$2es_\varepsilon R_\varepsilon\left[\left(q + \frac{1}{2}\right)\ln R_\varepsilon + \ln\left(15e^3\right)\right] = -\ln\left(\varepsilon^\beta + \varepsilon\right). \quad (18)$$

If $\varepsilon > 0$ is small enough then

$$m\left(B_{\varepsilon^\beta,R_\varepsilon}\right) \leq 2R_\varepsilon^{-q+\frac{1}{2}},$$

where $B_{\rho,R}$ is defined in Theorem 2.3.

We shall prove Lemma 3.1 in Section 4 (see also [8]). Our main result is

Theorem 3.2 *Let $s_0 > 0$, $\gamma \in (1,2)$, $\alpha \in (0,1)$, $\beta \in (0,1)$, $\alpha\beta < \frac{1}{4}$, $\nu = \frac{1}{4} + \alpha\beta$ and let $K \geq 1$, $M \geq 1$, $T > \frac{1}{2}$, $q > \frac{1}{2}$. Choosing $\varepsilon = n^{-\alpha}$, $\delta = n^{-\nu}$ and denoting*

$$f_n(x) = L_{\delta,g}(x; Y_1, \ldots, Y_n),$$

we have the estimate

$$\sup_{g\in\mathcal{G}_{s_0,\gamma,M,T}}\left(\sup_{f\in\mathcal{F}_{q,K}} E\left\|f_n-f\right\|^2_{L^2(\mathbb{R})}\right)$$
$$\leq C_3\left[\left(\frac{(s_0)^{\frac{1}{\gamma}}}{30\,(2q+1)\,e^4}\right)^{-\frac{q}{2}+\frac{1}{2}}(\ln(n^\alpha))^{\left(\frac{1}{2}-\frac{1}{2\gamma}\right)\left(-q+\frac{1}{2}\right)}+2(\ln(n^\alpha))^{\frac{2\nu-1}{\alpha}}\right]$$

for all $n\in\mathbb{N}$ large enough, where $C_3>0$ depends on q, K, T.

Proof. Let $f\in\mathcal{F}_{q,K}$ and $g\in\mathcal{G}_{s_0,\gamma,M,T}$. We have

$$\int_{|t|>R_\varepsilon}\left|f^{\mathrm{ft}}(t)\right|^2dt\leq\int_{|t|>R_\varepsilon}\frac{K\,dt}{(1+t^2)^q}\leq\int_{|t|>R_\varepsilon}\frac{K\,dt}{|t|^{2q}}\leq\frac{2K}{2q-1}R_\varepsilon^{-q+\frac{1}{2}}$$

for all $\varepsilon>0$ small enough. Combining this with Theorem 2.3 and Lemma 3.1, we get

$$\begin{aligned}\mathbb{E}\left\|L_{\delta,g}-f\right\|^2_{L^2(\mathbb{R})}&\leq C_1\left[\left(2+\frac{2K}{2q-1}\right)R_\varepsilon^{-q+\frac{1}{2}}+\frac{\delta^2}{\varepsilon^{4\beta}}+\frac{1}{n\delta^2}\right]\\&\leq C_2\left(R_\varepsilon^{-q+\frac{1}{2}}+\frac{\delta^2}{\varepsilon^{4\beta}}+\frac{1}{n\delta^2}\right)\end{aligned}$$

for all $\varepsilon>0$ small enough, where $C_2=C_1\left(2+\frac{2K}{2q-1}\right)$. Moreover, for all $\varepsilon>0$ small enough, from (25) we have the estimate

$$R_\varepsilon^{-q+\frac{1}{2}}\leq\left(\frac{(s_0)^{\frac{1}{\gamma}}}{30\,(2q+1)\,e^4}\right)^{-\frac{q}{2}+\frac{1}{2}}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\left(\frac{1}{2}-\frac{1}{2\gamma}\right)\left(-q+\frac{1}{2}\right)}.$$

Therefore

$$\mathbb{E}\left\|L_{\delta,g}-f\right\|^2_{L^2(\mathbb{R})}\leq C_2\left[\left(\frac{(s_0)^{\frac{1}{\gamma}}}{30\,(2q+1)\,e^4}\right)^{-\frac{q}{2}+\frac{1}{2}}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\left(\frac{1}{2}-\frac{1}{2\gamma}\right)\left(-q+\frac{1}{2}\right)}+\frac{\delta^2}{\varepsilon^{4\beta}}+\frac{1}{n\delta^2}\right].$$

Replacing $\varepsilon=n^{-\alpha}$, $\delta=n^{-\nu}$ to the right hand side of the latter inequality, we obtain

$$\mathbb{E}\left\|f_n-f\right\|^2_{L^2(\mathbb{R})}\leq C_2\left[\left(\frac{(s_0)^{\frac{1}{\gamma}}}{30\,(2q+1)\,e^4}\right)^{-\frac{q}{2}+\frac{1}{2}}(\ln(n^\alpha))^{\left(\frac{1}{2}-\frac{1}{2\gamma}\right)\left(-q+\frac{1}{2}\right)}+2n^{2\nu-1}\right]$$

for all $n\in\mathbb{N}$ large enough.

Furthermore, we have $n^{2\nu-1}=(n^\alpha)^{\frac{2\nu-1}{\alpha}}\leq(\ln(n^\alpha))^{\frac{2\nu-1}{\alpha}}$. Hence,

$$\mathbb{E}\left\|f_n-f\right\|^2_{L^2(\mathbb{R})}$$
$$\leq C_2\left[\left(\frac{(s_0)^{\frac{1}{\gamma}}}{30\,(2q+1)\,e^4}\right)^{-\frac{q}{2}+\frac{1}{2}}(\ln(n^\alpha))^{\left(\frac{1}{2}-\frac{1}{2\gamma}\right)\left(-q+\frac{1}{2}\right)}+2(\ln(n^\alpha))^{\frac{2\nu-1}{\alpha}}\right]$$

for all $n\in\mathbb{N}$ large enough.

Because $f \in \mathcal{F}_{q,K}$ and $g \in \mathcal{G}_{s_0,\gamma,M,T}$, we have

$$C_1 = \frac{1}{2\pi} \max\left\{1; \left\|f^{\mathrm{ft}}\right\|^2_{L^2(\mathbb{R})}; \left\|g^{\mathrm{ft}}\right\|^2_{L^2(\mathbb{R})}\right\} \leq \frac{1}{2\pi} \max\left\{1; 2\pi \int_{-\infty}^{+\infty} \frac{K dt}{(1+t^2)^q}; 2\pi T\right\}.$$

So, for all $n \in \mathbb{N}$ large enough, we have

$$\begin{aligned} &\mathbb{E}\left\|f_n - f\right\|^2_{L^2(\mathbb{R})} \\ &\leq C_3 \left[\left(\frac{(s_0)^{\frac{1}{\gamma}}}{30\,(2q+1)\,e^4}\right)^{-\frac{q}{2}+\frac{1}{2}} (\ln(n^\alpha))^{\left(\frac{1}{2}-\frac{1}{2\gamma}\right)\left(-q+\frac{1}{2}\right)} + 2(\ln(n^\alpha))^{\frac{2\nu-1}{\alpha}}\right], \end{aligned} \tag{19}$$

where

$$C_3 = \frac{1}{2\pi}\left(2 + \frac{2K}{2q-1}\right) \max\left\{1; 2\pi \int_{-\infty}^{+\infty} \frac{K dt}{(1+t^2)^q}; 2\pi T\right\}.$$

We note that the right hand side of (19) is independent of f and g. Therefore,

$$\begin{aligned} &\sup_{g \in \mathcal{G}_{s_0,\gamma,M,T}} \left(\sup_{f \in \mathcal{F}_{q,K}} \mathbb{E}\left\|f_n - f\right\|^2_{L^2(\mathbb{R})}\right) \\ &\qquad \leq C_3 \left[\left(\frac{(s_0)^{\frac{1}{\gamma}}}{30\,(2q+1)\,e^4}\right)^{-\frac{q}{2}+\frac{1}{2}} (\ln(n^\alpha))^{\left(\frac{1}{2}-\frac{1}{2\gamma}\right)\left(-q+\frac{1}{2}\right)} + 2(\ln(n^\alpha))^{\frac{2\nu-1}{\alpha}}\right] \end{aligned}$$

The proof of the theorem is completed. ∎

In case the error density function g is compactly supported, we get the following result

Theorem 3.3 *Let assumptions be as in Theorem 3.2. Moreover, assume that the density function g has $\operatorname{supp} g \subset [-L; L]$ where $\operatorname{supp} g$ is the support of g. Then*

$$\begin{aligned} &\sup_{g \in \mathcal{G}_{s_0,\gamma,M,T}} \left(\sup_{f \in \mathcal{F}_{q,K}} \mathbb{E}\left\|f_n - f\right\|^2_{L^2(\mathbb{R})}\right) \\ &\qquad \leq C_3 \left[\left(\frac{1}{30L\,(2q+1)\,e^4}\right)^{-\frac{q}{2}+\frac{1}{4}} (\ln(n^\alpha))^{-\frac{q}{2}+\frac{1}{4}} + 2(\ln(n^\alpha))^{\frac{2\nu-1}{\alpha}}\right], \end{aligned}$$

where constant $C_3 > 0$ depends on q, K, T.

Proof. For each $\varepsilon > 0$, from the definition of s_ε in (17), we get $\int_{|t| \geq s_\varepsilon} |g(t)|\, dt \leq \varepsilon$. Moreover, from the property of infimum, we have $\int_{|t| \geq s_\varepsilon - \eta} |g(t)|\, dt > \varepsilon$ for all $\eta > 0$, which implies $\int_{|t| \geq s_\varepsilon} |g(t)|\, dt \geq \varepsilon$. So

$$\int_{|t| \geq s_\varepsilon} |g(t)|\, dt = \varepsilon.$$

If $s_\varepsilon > L$ then $\int_{|t| \geq s_\varepsilon} |g(t)|\, dt = 0$ which is a contradiction. So $s_\varepsilon \leq L$ for all $\varepsilon > 0$ small enough.

From (21), for all $\varepsilon > 0$ small enough, we have

$$R_\varepsilon^2 \geq \frac{1}{15L\,(2q+1)\,e^4}\left[\frac{1}{\beta}\ln\left(\frac{1}{\varepsilon}\right) + \ln\left(\frac{1}{1+\varepsilon^{1-\beta}}\right)\right] \geq \frac{1}{30L\,(2q+1)\,e^4}\ln\left(\frac{1}{\varepsilon}\right).$$

It follows that

$$R_\varepsilon^{-q+\frac{1}{2}} \leq \left(\frac{1}{30L\,(2q+1)\,e^4}\right)^{-\frac{q}{2}+\frac{1}{4}}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-\frac{q}{2}+\frac{1}{4}}.$$

Therefore, combining with the proof of Theorem 3.2, we derive

$$\mathbb{E}\left\|L_{\delta,g} - f\right\|^2_{L^2(\mathbb{R})} \leq C_3\left[\left(\frac{1}{30L\,(2q+1)\,e^4}\right)^{-\frac{q}{2}+\frac{1}{4}}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-\frac{q}{4}+\frac{1}{4}} + \frac{\delta^2}{\varepsilon^{4\beta}} + \frac{1}{n\delta^2}\right].$$

Replacing $\varepsilon = n^{-\alpha}$, $\delta = n^{-\nu}$, we get

$$\mathbb{E}\left\|f_n - f\right\|^2_{L^2(\mathbb{R})} \leq C_3\left[\left(\frac{1}{30L\,(2q+1)\,e^4}\right)^{-\frac{q}{2}+\frac{1}{4}}(\ln(n^\alpha))^{-\frac{q}{2}+\frac{1}{4}} + 2(\ln(n^\alpha))^{\frac{2\nu-1}{\alpha}}\right]$$

for all $n \in \mathbb{N}$ large enough. The proof of the theorem is completed. ■

Thus, from Theorem 3.2 we see that the MISE of our estimator attains the logarithmic rate. The following theorem will show that the logarithmic rates are unavoidable.

Theorem 3.4 *Let $s_0 > 0$, $\gamma \in (1,2)$, $K \geq 1$, $M \geq 1$, $T > \frac{1}{2}$, $q > \frac{1}{2}$ and m is an integer greater than q. Then, for all $\delta > 0$ small enough, we have*

$$\sup_{g\in\mathcal{G}_{s_0,\gamma,M,T}}\left(\sup_{f\in\mathcal{F}_{q,K}}\mathbb{E}\left\|L_{\delta,g} - f\right\|^2_{L^2(\mathbb{R})}\right) \geq \frac{4^{-m}}{8\pi m}\left(\ln\left(\frac{1}{\delta}\right)\right)^{-2m}.$$

Proof. We consider the density function $g_0(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. The function $g_0 \in \mathcal{G}_{s_0,\gamma,M,T}$ and has $g_0^{\mathrm{ft}}(t) = e^{-\frac{t^2}{2}}$. With the density function $\psi(x) = \frac{1}{2}e^{-|x|}$ of Laplace distribution (also known as double-exponential density; See page 35, Section 2.4 in [2]), the function

$$f_0 = \underbrace{\psi * \psi * \ldots * \psi}_{m\,\text{times}}$$

is also a density function. This function has $f_0^{\mathrm{ft}}(t) = \left(\psi^{\mathrm{ft}}(t)\right)^m = \frac{1}{(1+t^2)^m}$. Since

$$\left|f_0^{\mathrm{ft}}(t)\right|^2 = \frac{1}{(1+t^2)^{2m}} \leq \frac{1}{(1+t^2)^m} \leq \frac{K}{(1+t^2)^q},$$

we have $f_0 \in \mathcal{F}_{q,K}$. We denote

$$H_\delta = \sup_{g\in\mathcal{G}_{s_0,\gamma,M,T}}\left(\sup_{f\in\mathcal{F}_{q,K}}\mathbb{E}\left\|L_{\delta,g} - f\right\|^2_{L^2(\mathbb{R})}\right).$$

From the equality (13) of Lemma 2.1, for all $f \in \mathcal{F}_{q,K}$ and $g \in \mathcal{G}_{s_0,\gamma,M,T}$, we have

$$\mathbb{E}\left\|L_{\delta,g} - f\right\|^2_{L^2(\mathbb{R})} \geq \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|f^{\text{ft}}(t)\right|^2 \frac{\delta^2}{\left(\delta + \left|g^{\text{ft}}(t)\right|^2\right)^2}dt.$$

Therefore,

$$\begin{aligned}
H_\delta &\geq \sup_{f\in\mathcal{F}_{q,K}} \mathbb{E}\left\|L_{\delta,g_0} - f\right\|^2_{L^2(\mathbb{R})} \geq \mathbb{E}\left\|L_{\delta,g_0} - f_0\right\|^2_{L^2(\mathbb{R})} \\
&\geq \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|f_0^{\text{ft}}(t)\right|^2 \frac{\delta^2}{\left(\delta + \left|g_0^{\text{ft}}(t)\right|^2\right)^2}dt. \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\delta^2}{\left(1+t^2\right)^{2m}\left(\delta + e^{-t^2}\right)^2}dt \\
&\geq \frac{1}{2\pi}\int_{e^{-t^2}\leq\delta} \frac{\delta^2}{\left(1+t^2\right)^{2m}\left(\delta + e^{-t^2}\right)^2}dt \\
&= \frac{1}{\pi}\int_{\sqrt{\ln\left(\frac{1}{\delta}\right)}}^{+\infty} \frac{\delta^2}{\left(1+t^2\right)^{2m}\left(\delta + e^{-t^2}\right)^2}dt \\
&\geq \frac{1}{4\pi}\int_{\sqrt{\ln\left(\frac{1}{\delta}\right)}}^{+\infty} \frac{1}{\left(1+t^2\right)^{2m}}dt \\
&\geq \frac{1}{4\pi}\int_{\sqrt{\ln\left(\frac{1}{\delta}\right)}}^{+\infty} \frac{2t}{\left(1+t^2\right)^{2m+1}}dt.
\end{aligned}$$

By direct computations, we derive

$$H_\delta \geq \frac{1}{8\pi m}\left(1 + \ln\left(\frac{1}{\delta}\right)\right)^{-2m} \geq \frac{4^{-m}}{8\pi m}\left(\ln\left(\frac{1}{\delta}\right)\right)^{-2m}$$

for all $\delta > 0$ small enough. The proof of the theorem is completed. ■

Choosing $\delta = n^{-\nu}$ with $\nu = \frac{1}{4} + \alpha\beta$, $\alpha \in (0,1)$, $\beta \in (0,1)$, $\alpha\beta < \frac{1}{4}$ as in Theorem 3.2 and denoting $f_n(x) = L_{\delta,g}(x; Y_1, \ldots, Y_n)$, we get

Corollary 3.5 *Let assumptions be as in Theorem 3.2 and Theorem 3.4. Then*

$$\sup_{g\in\mathcal{G}_{s_0,\gamma,M,T}}\left(\sup_{f\in\mathcal{F}_{q,K}} \mathbb{E}\left\|f_n - f\right\|^2_{L^2(\mathbb{R})}\right) \geq \frac{4^{-m}}{8\pi m}\left(\nu\ln(n)\right)^{-2m}$$

for all $n \in \mathbb{N}$ large enough.

4 Proof of Lemma 3.1

To estimate the Lebesgue measure of low level sets, we shall use the following result (see Theorem 4, Section §11.3 in [7]).

Lemma 4.1 *Let $f(z)$ be an analytic function in the disk $\{z : |z| \leq 2eR\}$, $|f(0)| = 1$, and let η be an arbitrary small positive number. Then the estimate*

$$\ln |f(z)| > -\ln\left(\frac{15e^3}{\eta}\right). \ln M_f(2eR)$$

is valid everywhere in the disk $\{z : |z| \leq R\}$ except a set of disks (D_j) with sum of radius $\sum r_j \leq \eta R$, where $M_f(r) = \max_{|z|=r} |f(z)|$.

Using the latter lemma, we now state and prove an estimate for low level sets.

Theorem 4.2 *Let the density function g be in $\mathcal{G}_{s_0,\lambda,M,T}$ where $s_0 > 0$, $\lambda \in (1,2)$, $M \geq 1$, $T > \frac{1}{2}$ and let $\beta \in (0,1)$, $q > \frac{1}{2}$. For $\varepsilon > 0$ small enough, we choose s_ε as in (17) and choose R_ε to satisfy*

$$2es_\varepsilon R_\varepsilon\left[\left(q+\frac{1}{2}\right)\ln R_\varepsilon + \ln\left(15e^3\right)\right] = -\ln\left(\varepsilon^\beta + \varepsilon\right). \tag{20}$$

Then

$$\lim_{\varepsilon\to 0} R_\varepsilon = +\infty.$$

Moreover, if ε is small enough, we have

$$m\left(D_{\varepsilon^\beta+\varepsilon}\right) \leq 2R_\varepsilon^{-q+\frac{1}{2}},$$

where

$$D_{\varepsilon^\beta+\varepsilon} = \left\{z \in \mathbb{R} : |\Phi_\varepsilon(z)| < \varepsilon^\beta + \varepsilon,\ |z| < R_\varepsilon\right\}$$

with

$$\Phi_\varepsilon(z) = \int_{-s_\varepsilon}^{s_\varepsilon} g(t)\, e^{zt} dt, \quad z \in \mathbb{C}.$$

Proof. The proof of the theorem is divided into 2 steps. In Step 1, we give the existence of R_ε and prove that $\lim_{\varepsilon\to 0} R_\varepsilon = +\infty$. In Step 2, we shall estimate $m\left(D_{\varepsilon^\beta+\varepsilon}\right)$.

Step 1. We consider the function

$$\psi(R) = 2es_\varepsilon R\left[\left(q+\frac{1}{2}\right)\ln R + \ln\left(15e^3\right)\right] + \ln\left(\varepsilon^\beta + \varepsilon\right), \quad R \geq 0.$$

We have $\psi(R) \to +\infty$ as $R \to +\infty$ and $\psi(R) \to \ln\left(\varepsilon^\beta + \varepsilon\right) < 0$ as $R \to 0$ for $\varepsilon > 0$ small enough. So there exists an $R_\varepsilon > 0$ such that $\psi(R_\varepsilon) = 0$, i.e, R_ε satisfies (20). Also from (20), we get

$$(2q+1)\, eR_\varepsilon \ln\left(15e^3 R_\varepsilon\right) \geq \frac{1}{s_\varepsilon}\ln\left(\frac{1}{\varepsilon^\beta + \varepsilon}\right).$$

In view of the inequality $\ln x \leq x$ for all $x > 0$, we have

$$R_\varepsilon^2 \geq \frac{1}{15e^4(2q+1)}\frac{\frac{1}{\beta}\ln\left(\frac{1}{\varepsilon}\right) + \ln\left(\frac{1}{1+\varepsilon^{1-\beta}}\right)}{s_\varepsilon}. \tag{21}$$

From the definition of s_ε in (17), we get $\int_{|t|\geq s_\varepsilon} |g(t)|\, dt = \varepsilon$. Thus

$$Me^{-s_0(s_\varepsilon)^\gamma} \geq e^{-s_0(s_\varepsilon)^\gamma} \int_{-\infty}^{+\infty} e^{s_0|t|^\gamma} |g(t)|\, dt \geq e^{-s_0(s_\varepsilon)^\gamma} \int_{|t|\geq s_\varepsilon} e^{s_0|t|^\gamma} |g(t)|\, dt \geq \varepsilon.$$

The latter inequality implies

$$s_\varepsilon \leq \left(\frac{1}{s_0}\right)^{\frac{1}{\gamma}} \left(\ln\left(\frac{M}{\varepsilon}\right)\right)^{\frac{1}{\gamma}}. \tag{22}$$

From (21) and (22), we get

$$R_\varepsilon^2 \geq \frac{1}{15e^4(2q+1)} \left[\frac{(s_0)^{\frac{1}{\gamma}}}{\beta} \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\left(\ln\left(\frac{M}{\varepsilon}\right)\right)^{\frac{1}{\gamma}}} + (s_0)^{\frac{1}{\gamma}}. \frac{\ln\left(\frac{1}{1+\varepsilon^{1-\beta}}\right)}{\left(\ln\left(\frac{M}{\varepsilon}\right)\right)^{\frac{1}{\gamma}}}\right]. \tag{23}$$

As

$$\lim_{\varepsilon\to 0} \left[\frac{\ln\left(\frac{1}{\varepsilon}\right)}{\left(\ln\left(\frac{M}{\varepsilon}\right)\right)^{\frac{1}{\gamma}}} : \left(\ln\left(\frac{M}{\varepsilon}\right)\right)^{\frac{1}{2}-\frac{1}{2\gamma}}\right] = +\infty,$$

we obtain

$$\frac{\ln\left(\frac{1}{\varepsilon}\right)}{\left(\ln\left(\frac{M}{\varepsilon}\right)\right)^{\frac{1}{\gamma}}} \geq \left(\ln\left(\frac{M}{\varepsilon}\right)\right)^{\frac{1}{2}-\frac{1}{2\gamma}}, \tag{24}$$

for all $\varepsilon > 0$ small enough. Furthermore, since

$$\lim_{\varepsilon\to 0} \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\left(\ln\left(\frac{M}{\varepsilon}\right)\right)^{\frac{1}{\gamma}}} = +\infty, \quad \lim_{\varepsilon\to 0} \frac{\ln\left(\frac{1}{1+\varepsilon^{1-\beta}}\right)}{\left(\ln\left(\frac{M}{\varepsilon}\right)\right)^{\frac{1}{\gamma}}} = 0,$$

we have

$$R_\varepsilon^2 \geq \frac{(s_0)^{\frac{1}{\gamma}}}{30(2q+1)\beta e^4} \left(\ln\left(\frac{M}{\varepsilon}\right)\right)^{\frac{1}{2}-\frac{1}{2\gamma}} \geq \frac{(s_0)^{\frac{1}{\gamma}}}{30(2q+1)e^4} \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{1}{2}-\frac{1}{2\gamma}} \tag{25}$$

for all $\varepsilon > 0$ small enough.

From the inequality (25), we obtain $R_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

Step 2. We see that Φ_ε is an entire function, i.e, a complex function analytic on $\mathbb{C}$, because

$$|\Phi_\varepsilon(z)| = \left|\int_{-s_\varepsilon}^{s_\varepsilon} g(t)\, e^{zt} dt\right| \leq \int_{-s_\varepsilon}^{s_\varepsilon} |g(t)|\, e^{|z||t|} dt \leq e^{|z|s_\varepsilon}, \quad z \in \mathbb{C}. \tag{26}$$

Since g is a density function, the function g^{ft} is a non-trivial function of $L^1(\mathbb{R})$. Hence, Φ_ε is a non-trivial entire function, and so there exists an $x_0 \in \mathbb{R}$ such that $|\Phi_\varepsilon(x_0)| = C_4 > 0$. Changing variable if necessary, we may assume that $|\Phi_\varepsilon(0)| = 1$ if $\varepsilon > 0$ small enough.

For all $|z| = 2eR_\varepsilon$, from (26), we get $|\Phi_\varepsilon(z)| \leq e^{2es_\varepsilon R_\varepsilon}$ and that

$$\ln M_{\Phi_\varepsilon}(2eR_\varepsilon) \equiv \ln\left(\max_{|z|=2eR_\varepsilon} |\Phi_\varepsilon(z)|\right) \leq 2es_\varepsilon R_\varepsilon.$$

We choose $\eta_\varepsilon = R_\varepsilon^{-q-\frac{1}{2}}$. Then, for all $|z| \leq R_\varepsilon$, applying Lemma 4.1, we have the estimate

$$\begin{aligned} |\Phi_\varepsilon(z)| &\geq \exp\left\{-\ln\left(\frac{15e^3}{R_\varepsilon^{-q-\frac{1}{2}}}\right)\ln M_{\Phi_\varepsilon}(2eR_\varepsilon)\right\} \\ &\geq \exp\left\{-2es_\varepsilon R_\varepsilon\left[\left(q+\frac{1}{2}\right)\ln R_\varepsilon + \ln\left(15e^3\right)\right]\right\} \\ &= \varepsilon^\beta + \varepsilon \end{aligned}$$

except a set of disks $\{D(z_j, r_j)\}_{j\in J}$ whose sum of radius is less than $\eta_\varepsilon R_\varepsilon = R_\varepsilon^{-q+\frac{1}{2}}$. This implies

$$D_{\varepsilon^\beta+\varepsilon} \equiv \left\{z \in \mathbb{R} : |\Phi_\varepsilon(z)| < \varepsilon^\beta + \varepsilon,\ |z| < R_\varepsilon\right\} \subset \bigcup_{j\in J}(D(z_j, r_j)\cap\mathbb{R}) \tag{27}$$

where we recall

$$D(z_j, r_j) = \{z \in \mathbb{C} : |z - z_j| < r_j\}, \quad j \in J.$$

From (27) we derive

$$m(D_{\varepsilon^\beta+\varepsilon}) \leq m\left(\bigcup_{j\in J}(D(z_j, r_j)\cap\mathbb{R})\right) \leq \sum_{j\in J} m(D(z_j, r_j)\cap\mathbb{R}) \leq \sum_{j\in J} 2r_j \leq 2R_\varepsilon^{-q+\frac{1}{2}}$$

for all $\varepsilon > 0$ small enough. This completes the proof of Step 2 and the proof of our theorem. ■

Finally, we turn to the

Proof of Lemma 3.1

For each $\varepsilon > 0$, we put

$$g_\varepsilon(t) = \begin{cases} g(t), & |t| \leq s_\varepsilon, \\ 0, & |t| > s_\varepsilon. \end{cases}$$

We see that $g_\varepsilon^{\text{ft}}(x) = \Phi_\varepsilon(ix)$ if $x \in \mathbb{R}$. For all $x \in \mathbb{R}$, we have

$$\left|g_\varepsilon^{\text{ft}}(x) - g^{\text{ft}}(x)\right| = \left|\int_{-s_\varepsilon}^{s_\varepsilon} g(t)e^{itx}dt - \int_{-\infty}^{+\infty} g(t)e^{itx}dt\right| \leq \int_{|t|\geq s_\varepsilon}|g(t)|\,dt = \varepsilon.$$

Thus, if $\left|g^{\text{ft}}(x)\right| < \varepsilon^\beta$, $|x| < R_\varepsilon$ then $\left|g_\varepsilon^{\text{ft}}(x)\right| < \varepsilon^\beta + \varepsilon$. This implies $|\Phi_\varepsilon(ix)| < \varepsilon^\beta + \varepsilon$. Applying Theorem 4.2, we get

$$m\left(B_{\varepsilon^\beta, R_\varepsilon}\right) \leq m(D_{\varepsilon^\beta+\varepsilon}) \leq 2R_\varepsilon^{-q+\frac{1}{2}}$$

for all $\varepsilon > 0$ small enough. The proof of the lemma is completed. ■

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Dang Duc Trong, Faculty of Mathematics and Computer Science, Ho Chi Minh City National University, 227 Nguyen Van Cu, District 5, Ho Chi Minh City, Viet Nam.

Email address: ddtrong@hcmus.edu.vn

Cao Xuan Phuong, Faculty of Mathematics and Statistics, Ton Duc Thang University, Nguyen Huu Tho, District 7, Ho Chi Minh City, Viet Nam.

Email address: caoxuanphuong@tdt.edu.vn

Truong Trung Tuyen, Department of Mathematics, Indiana University, Bloomington IN 47405, USA.

Email address: truongt@indiana.edu

Dinh Ngoc Thanh, Faculty of Mathematics and Computer Science, Ho Chi Minh City National University, 227 Nguyen Van Cu, District 5, Ho Chi Minh City, Viet Nam.

Email address: dnthanh@hcmus.edu.vn