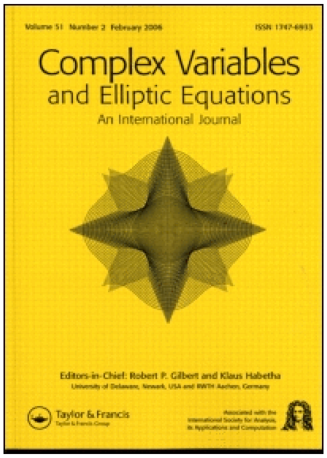

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The growth at infinity of a sequence of entire functions of bounded orders

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In this article, we shall consider the growth at infinity of a sequence (P_n) of entire functions of bounded orders. Our results extend a result of J. Muller and A. Yavrian for the growth near infinity of a polynomial sequence. Given a sequence of entire functions of bounded orders $P_n(z)$, we found a nearly optimal condition, given in terms of zeros of P_n, for which (k_n) we have

$$\limsup_{n\to\infty} \|P_n\|_K^{1/k_n} \leq 1$$

for all compact sets $K \subset \mathbb{C}$ (Theorem 3). Exploring the growth of a sequence of entire functions of bounded orders lead naturally to an extremal function which is similar to the Siciak's extremal function (Section 6).

Keywords: capacity; entire functions of bounded orders; geometric rate growth; non-thin set

AMS Subject Classifications: 30C85; 30D15; 31A15

1. Introduction and main results

Let $P(z)$: $\mathbb{C} \to \mathbb{C}$ be an entire function. We recall that (see Lecture 1 in [1]): if

$$P(z) = \sum_{n=0}^{\infty} a_n z^n$$

then its order ρ is

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)}.$$

An entire function is called as genus zero if its order is less than 1.

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If $P(z)$ is an entire function of finite order ρ, by Hadamard factorization theorem (see Theorem 1, p. 26 in [1]), $P(z)$ has a representation

$$P(z) = az^m e^{W_q(z)} \prod_j G\left(\frac{z}{z_j}, p\right) \tag{1.1}$$

where $p = [\rho]$ the integer part of ρ, $W_q(z)$ is a polynomial of degree $q \leq \rho$ with $W_q(0) = 0$, $z_j's$ are zeros of $P(z)$ different from 0, and

$$G(z, p) = (1 - z)\exp\left\{z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right\}.$$

For an entire function $P(z)$ of order ρ with representation (1.1) we define its degree $d^*(P)$ by

$$d^*(P) = m + \sup_{|z|\leq 1} |W_q(z)| + \sum_{|z_j|\leq 1} \frac{1}{|z_j|^p} + \sum_{|z_j|>1} \frac{1}{|z_j|^{p+1}}. \tag{1.2}$$

Remark If $P(z)$ is an entire function of genus zero then in the representation (1.1) we have $q = p = 0$. If $P(z)$ is a polynomial then $d^*(P) \leq \deg(P)$ where $\deg(P)$ is the usual degree of $P(z)$.

The growth at infinity of entire functions is a topic of great concernment. Let (P_n) be a sequence of entire functions and let (k_n) be a sequence of positive numbers. It is an interesting question that for which sequences (P_n) and (k_n) we have

$$\limsup_{n\to\infty} \|P_n\|_K^{1/k_n} \leq 1$$

for all compact sets $K \subset \mathbb{C}$, where

$$\|P_n\|_K = \sup_{z\in K} |P_n(z)|.$$

This question arises naturally as we want to show that a series

$$s(z) = \sum_{n=1}^{\infty} P_n(z) \tag{1.3}$$

converges locally uniform to an entire function. Usually we want to find a sequence of positive numbers $k_n \geq n$ such that

$$\limsup_{n\to\infty} \|P_n\|_K^{1/k_n} < 1$$

for all compact sets $K \subset \mathbb{C}$. Of course, if the series (1.3) converges locally uniform then for any sequence (k_n) that diverges to ∞ we have

$$\limsup_{n\to\infty} \|P_n\|_K^{1/k_n} \leq 1 \tag{1.4}$$

for all compact sets $K \subset \mathbb{C}$. If we choose the sequence (k_n) very large then it is most likely that the LHS of (1.4) equals 1, so we cannot conclude that (1.3) converges, if we choose (k_n) very small then we may not know whether the LHS of (1.4) is bounded or not (in particular, when we want to show that if $s(z)$ converges on a certain subset of $\mathbb{C}$ then it converges on $\mathbb{C}$). Hence, it is worthwhile to consider what conditions should any 'generic' sequence (k_n) satisfy if it satisfies (1.4). Here 'generic' means that the constant C_0^* defined in Section 2 is not zero. (For the case $C_0^* = 0$ we cannot hope to have any conclusion about the sequence (k_n), because if $P_n(z)$ converges uniformly to 0 then any bounded sequence k_n will satisfy (1.4). We state this question as

Question 1 Given a sequence of entire functions (P_n). Assume that (k_n) is a sequence of positive numbers such that (1.4) is satisfied. What can we say about k_n in terms of zeros of P_n's?

In [2], the authors gained interesting results which combine the growth of a sequence of polynomials on a 'small subset' of $\mathbb{C}$ with the growth of itself on the whole plane. The 'small subsets' as mentioned are non-thin. We recall that [2] a domain G, with ∂G having positive capacity, is non-thin at its boundary point $\zeta \in \partial G$ (or ζ is a regular point of G) if and only if

$$\lim_{z \in G, z \to \zeta} g(z, w) = 0$$

for all $w \in G$ where $g(\cdot,\cdot)$ is the Green function of G. One of the main results in [2] is stated as below (see Lemma 2 in [2])

PROPOSITION 1 *Let (k_n) be a sequence of positive numbers and let (P_n) be a sequence of polynomials satisfying* $\deg(P_n) \le k_n$. If $E \subseteq \mathbb{C}$ *is closed and non-thin at* ∞ *so that*

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \le 1, \quad \textit{for all } z \in E,$$

then

$$\limsup_{n\to\infty} \|P_n\|_R^{1/k_n} \le 1, \quad \textit{for all } R > 0,$$

where $\|P_n\|_R = \sup\{|P_n(z)| : |z| \le R\}$.

Saying roughly, Proposition 1 states that if E is a set being non-thin at infinity and (P_n) is a sequence of polynomials having a geometric rate growth at each point in E then (P_n) has a geometric rate growth in $\mathbb{C}$ also. Here by geometric rate growth we refer to one property that the sequence of polynomials in Proposition 1 satisfies: moreover, if we have that (k_n) is an increasing sequence of integer numbers, and

$$\limsup_{n\to\infty} \|P_n\|_K^{1/k_n} < 1$$

for any compact set $K \subseteq \mathbb{C}$, then the summation $\sum_{n=1}^{\infty} P_n(z)$ converges with locally geometric rate. Hence for the sake of abbreviation, we use the term geometric rate growth to call any sequence of entire functions (P_n) satisfying the condition $\limsup_{R\to\infty} \|P_n\|_R^{1/k_n} \le 1$ for some sequence $(k_n > 0)$ and for any $R > 0$ (see also the discussion in the previous paragraph and Question 1).

There are two interesting questions rising from this result.

Question 2 Does the conclusion of Proposition 1 still hold if P_n are non-polynomial entire functions?

Question 3 If (P_n) has a non-geometric rate growth in E, i.e. if instead of the condition

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \le 1, \quad \text{for all } z \in E,$$

we require only

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \le h(|z|), \quad \text{for all } z \in E,$$

where h is not necessarily a bounded function, does (P_n) remain the same rate of growth in $\mathbb{C}$?

Remark 1 The sequences (P_n) whose growth is non-geometric arises naturally in practice. For a simple example, we can consider the case when $P_n(z) = z^n$ and $k_n = n$. This sequence does not have the geometric rate growth, however $|P_n(z)|^{1/k_n}$ grows like $|z|$.

In this article, we give some partial answers to these three questions when the sequence (P_n) is of bounded orders: that is $\rho_n \le \rho$ for all $n \in \mathbb{N}$ where ρ_n is the order of P_n, and $\rho > 0$ is a constant. We obtain more satisfying results for the case (P_n) is a sequence of entire function of genus zero (Sections 2–5), and less satisfying results for the general case when P_n are of bounded orders (Section 6).

For Question 1 we obtain a nearly optimal answer (Theorem 3).

For Question 3, the answer is confirmation in the case $P_n(z)$ is of genus zero, $h(z)$ is a polynomial and E is a closed set satisfying

$$\limsup_{R\to\infty} \frac{\log cap(E_R)}{\log R} = \beta > 0,$$

where

$$E_R = E \cap \{z : |z| \le R\}$$

(Theorem 2). As a corollary we immediately get that E must be non-thin at infinity. This result was proved in [2] using Wiener's criterion.

The main tools that we used to get our results are: the degree of entire functions of finite orders; an equivalent relation between two different kinds of growth, that is the modulus growth and the logarithmic-integration growth, of a sequence of entire functions of genus zero (Lemmas 2 and 3); the Weierstrass inequality (Lemma 1); and the integral representation of the Green's function (Theorem 2).

Exploring the growth of a sequence of entire functions of bounded orders naturally leads us to consider an extremal function which is analogous to the Siciak's extremal function (Section 6). In [3], the authors also defined a similar extremal function for other classes of functions. They also define a 'degree' for continuous functions $P(z)$ but based on a filtration of the space of continuous functions rather than based on the zeros of the function $P(z)$ as our treatment here. Their degree is always an integer while our degree may be any positive number.

In a recent article of Truong, some of the results in this article were extended to the case of entire functions of several complex variables [4]. Various results in the case of one complex variable (see, e.g. [5,6]) can hopefully be extended to this general case of several complex variables.

This article consists of six sections. In Sections 2 and 3, we consider the growth of a sequence of entire functions of genus zero. In Sections 4 and 5, we consider some consequences, refinements and examples (including grouped power series, grouped Fourier series and Fourier transform). In Section 6, we discuss the growth of a sequence of entire functions of bounded orders, and define an extremal function that is naturally related to the problem.

2. Notations and lemmas

For an entire function f we use notations

$$\|f\|_R = \sup_{\{|z|\le R\}} |f(z)|$$

$$C(f,R) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|f(\mathrm{Re}^{it})|\right\},$$

$$\eta(f,R) = \text{ the number of elements of } \{z : 0 < |z| \le R, f(z) = 0\}.$$

Hereafter (in particular, in Sections 2–4), unless specified otherwise, we always consider a sequence of positive numbers (k_n) and a sequence of entire functions of genus zero (P_n) having the following form

$$P_n(z) = a_n z^{\alpha_n} \prod_j (1 - z/z_{n,j}). \tag{2.1}$$

We put

$$C_0 = \limsup_{n\to\infty} C(P_n, 1)^{1/k_n},$$

$$C_0^* = \liminf_{n\to\infty} C(P_n, 1)^{1/k_n},$$

$$\eta(R) = \limsup_{n\to\infty} \frac{\eta(P_n, R)}{k_n}.$$

For definition of capacity of a compact set, its properties and its relations to the Green's function and the harmonic measure of the set, one can refer to [7,12].

For convenience of the reader we recall here Weierstrass's inequality (see Lemma 15.8 in [8]).

LEMMA 1 *For* $p \in \mathbb{N} = \{1, 2, \ldots\}$ *define*

$$G(z,p) = (1-z)\exp\left\{z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right\}.$$

(*By convenience we define* $G(z,0) = 1 - z$). *Let* $z \in \mathbb{C}$.

(1) *If* $|z| \le 1$ *then*

$$|G(z,p)| \le \left|\exp\left\{\frac{z^{p+1}}{p+1}\right\}\right|(1 + O(|z^{p+2}|)).$$

(2) *If z is small enough then*

$$|G(z,p)| \sim \left|\exp\left\{\frac{z^{p+1}}{p+1}\right\}\right|(1+O(|z^{p+2}|)).$$

(3) *If* $|z|\geq 1$ *then*

$$|G(z,p)| \leq \exp\{\lambda_p|z|^p\} \leq \exp\{\lambda_p|z|^{p+1}\}$$

where

$$\lambda_p = 1+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p}.$$

Here $O(z)$ *usually means that there is a constant* $C>0$ *such that* $O(z)\leq C|z|$ *for z small enough.*

Proof The proof of 3 is easy. For proof of 1 and 2 we apply Lemma 15.8 in [8] to the function $G(z,p+1)$. ∎

Lemma 2

(i) *Let us assume that* $k_n \geq d^*(P_n)$ *for all* $n\in\mathbb{N}$. *If* $C_0=0$ *then for all* $R>0$

$$\limsup_{n\to\infty} \|P_n\|_R^{1/k_n} = 0.$$

(ii) *Assume that* $C_0^*>0$ *and*

$$\limsup_{n\to\infty} C(P_n,R)^{1/k_n} \leq h(R),$$

where h satisfies

$$\liminf_{R\to\infty}\frac{\log h(R)}{\log R} \leq \tau.$$

Then for all $R>0$ *we have* $\eta(R)\leq\tau$.

Proof (i) For each $n\in\mathbb{N}$, $z\in\mathbb{C}$, by applying Jensen's identity (see, e.g. Theorem 15.18 in [8]) gives

$$\begin{aligned}
|P_n(z)| &\leq |a_n||z|^{\alpha_n}\prod_j(1+R/|z_{n,j}|)\\
&\leq |a_n||z|^{\alpha_n}\prod_{|z_{n,j}|\leq 1}(1+R)/|z_{n,j}|\exp\left\{|z|\sum_{|z_{n,j}|>1}1/|z_{n,j}|\right\}\\
&= (|a_n||z|^{\alpha_n}\prod_{|z_{n,j}|\leq 1}1/|z_{n,j}|)(1+R)^{\eta(P_n,1)}\exp\left\{|z|\sum_{|z_{n,j}|>1}1/|z_{n,j}|\right\}\\
&= C(P_n,1)(1+R)^{\eta(P_n,1)}\exp\left\{|z|\sum_{|z_{n,j}|>1}1/|z_{n,j}|\right\}\\
&\leq C(P_n,1)(1+R)^{d^*(P_n)}\exp\{|z|d^*(P_n)\}.
\end{aligned}$$

Thus

$$\begin{aligned}\limsup_{n\to\infty} \|P_n\|_R^{1/k_n} &\le \limsup_{n\to\infty} C(P_n, 1)^{1/k_n}(1+R)^{d^*(P_n)/k_n} \exp\{|z|d^*(P_n)/k_n\}\\ &\le (1+R)\exp\{R\} \limsup_{n\to\infty} C(P_n, 1)^{1/k_n}\\ &= (1+R)\exp\{R\}C_0 = 0.\end{aligned}$$

(ii) Fix $R > 0$ and choose $s > R$. Applying Jensen's formula gives

$$C(P_n, R) = |a_n|R^{\alpha_n} \prod_j \max\left\{1, \frac{R}{|z_{n,j}|}\right\}.$$

We get

$$C(P_n, s) \ge C(P_n, R)(s/R)^{\eta(P_n,R)},$$

hence

$$\begin{aligned}h(s) &\ge \limsup_{n\to\infty} C(P_n, s)^{1/k_n}\\ &\ge \limsup_{n\to\infty} C(P_n, R)^{1/k_n}(s/R)^{\eta(P_n,R)/k_n}\\ &\ge \liminf_{n\to\infty} C(P_n, R)^{1/k_n} \limsup_{n\to\infty}(s/R)^{\eta(P_n,R)/k_n}\\ &= C_0^*(s/R)^{\eta(R)}.\end{aligned}$$

Thus

$$\begin{aligned}\tau &\ge \liminf_{s\to\infty} \frac{\log h(R)}{\log R}\\ &\ge \liminf_{s\to\infty} \frac{\log C_0^*(s/R)^{\eta(R)}}{\log s}\\ &= \eta(R).\end{aligned}$$

This completes the proof of Lemma 2. ∎

We end this section with a result relating the maximum and logarithm norms.

LEMMA 3 *Let us assume that* $k_n \ge d^*(P_n)$, $C_0 < \infty$, *and that*

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{\left|\sum_{|z_{n,j}|\ge R}(1/z_{n,j})\right|}{k_n} = 0, \tag{2.2}$$

and there exists a sequence (R_n) *of positive real numbers tending to* ∞ *such that*

$$\limsup_{n\to\infty} \frac{\eta(P_n, R_n)}{k_n} < \infty. \tag{2.3}$$

If

$$\liminf_{R\to\infty} \limsup_{n\to\infty} \frac{\log C(P_n, R)^{1/k_n}}{\log R} \le \tau,$$

then for all $R > 0$ we have

$$\limsup_{n\to\infty} \|P_n\|_R^{1/k_n} \le C_0(1+R)^\tau.$$

Proof Without loss of generality, we may assume that $R > 1$.

In view of Lemma 2(i) we only consider the case in which $C_0 > 0$.

To prove the result we need to show that: each subsequence of (P_n) contains a subsequence that satisfies the conclusion of Lemma 3. We choose a subsequence, still denoted by (P_n), such that

$$\lim_{n\to\infty} C(P_n, 1)^{1/k_n} = C_1.$$

We note that $C_1 \le C_0$.

If $C_1 = 0$ then we get the conclusion by Lemma 2(i).

If $C_1 > 0$ then applying Lemma 2 for this subsequence gives

$$\eta(R) \le \tau,$$

for all $R > 0$. Choosing a $\beta > 1$, we have

$$\|P_n\|_R \le |a_n| R^{\alpha_n} \prod_{|z_{n,j}|\le \beta R} (1 + R/|z_{n,j}|) \sup_{|z|\le R} \prod_{|z_{n,j}|>\beta R} |1 - z/z_{n,j}|.$$

Noting that for $|R/z_{n,j}| \le 1/\beta$, by Lemma 1 we have

$$|1 - z/z_{n,j}| = \left|\exp\left\{-\frac{z}{z_{n,j}}\right\}\right| \left(1 + O\left(\frac{|z|^2}{|z_{n,j}|^2}\right)\right) = \left|\exp\left\{-\frac{z}{z_{n,j}} + O\left(\frac{|z|}{\beta}\frac{1}{|z_{n,j}|}\right)\right\}\right|$$

with β large enough. Hence

$$\|P_n\|_R \le C(P_n, 1)(1+R)^{\eta(P_n,\beta R)} \sup_{|z|\le R} \left|\exp\left\{-\sum_{|z_{n,j}|\ge\beta R} \frac{z}{z_{n,j}} + \frac{R}{\beta}\sum_{|z_{n,j}|\ge \beta R} O\left(\frac{1}{|z_{n,j}|}\right)\right\}\right|.$$

It follows that

$$\begin{aligned}\limsup_{n\to\infty} \|P_n\|_R^{1/k_n} &\le C_1(1+R)^{\eta(R)} \limsup_{n\to\infty} \left|\exp\left\{\frac{1}{k_n}\left[-\sum_{|z_{n,j}|\ge\beta R}\frac{z}{z_{n,j}} + \frac{R}{\beta}\sum_{|z_{n,j}|\ge\beta R} O\left(\frac{1}{|z_{n,j}|}\right)\right]\right\}\right| \\ &\le C_0(1+R)^\tau \limsup_{n\to\infty} \left|\exp\left\{\frac{R}{\beta}\frac{1}{k_n}\sum_{|z_{n,j}|\ge\beta R} O\left(\frac{1}{|z_{n,j}|}\right)\right\}\right|.\end{aligned}$$

Letting β tend to ∞ we get

$$\limsup_{n\to\infty} \|P_n\|_R^{1/k_n} \le C_0(1+R)^\tau.$$

■

3. The growth of sequences of entire functions of genus zero

In this section, we still use the convenience of Section 2, that is (P_n) is a sequence of entire functions of genus zero with representation (2.1), and (k_n) is a sequence of positive numbers.

Theorems 1 and 2 answer Questions 2 and 3. In Theorem 3 we answer Question 1, whose results justify the conditions imposed in Theorems 1 and 2. In Theorem 4, we show how the conditions of Theorems 1 and 2 can be improved in view of Theorem 3.

THEOREM 1 *Let E be a closed set being non-thin at ∞. Let us assume that $k_n \geq d^*(P_n)$, and that*

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{|\sum_{|z_{n,j}|\geq R} 1/z_{n,j}|}{k_n} = 0, \tag{3.1}$$

and there exists a sequence $\{R_n\}$ of positive real numbers tending to ∞ such that

$$\limsup_{n\to\infty} \frac{\eta(P_n, R_n)}{k_n} < \infty. \tag{3.2}$$

If for each $z \in E$ one has

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1,$$

then

$$\limsup_{n\to\infty} \|P_n\|_R^{1/k_n} \leq 1, \quad \textit{for all } R > 0.$$

Remark As will be shown in Theorem 3, from the assumptions of Lemma 2(ii) and assumption $k_n \geq d^*(P_n)$, we shall obtain conditions (3.1) and (3.2).

Theorem 1 generalizes Proposition 1. To show this end, we note that if P_n are polynomials and $d_n \geq \deg(P_n)$ then

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{|\sum_{|z_{n,j}|\geq R} 1/z_{n,j}|}{d_n} \leq \lim_{R\to\infty} \frac{1}{R} = 0,$$

and for all $n \in \mathbb{N}$ and $R > 0$

$$\frac{\eta(P_n, R)}{d_n} \leq 1.$$

Proof For each compact set A in the complex plan, we take $g(A, z)$ to be its Green's function having pole at infinity of the unbounded component of $\mathbb{C} \setminus A$ and extend $g(A, z)$ to be zero outside that component.

For each $R > 0$ let E_R^* be the union of E_R with the bounded components of $\mathbb{C}\setminus E_R$. Then $E_R \subset E_R^*$ and $C\setminus E_R^*$ has no bounded components. For $z \in E_R^*$ we claim that

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1.$$

In fact, the compare principle gives $g(E_R^*, z) \leq g(E_R, z)$ for all $z \in \mathbb{C}$. Thus by Lemma 2 in [2], for all $s > 0$ we have

$$\lim_{R\to\infty}\int_0^{2\pi} g(E_R^*, se^{it})\mathrm{d}t \leq \lim_{R\to\infty}\int_0^{2\pi} g(E_R, se^{it})\mathrm{d}t = 0.$$

Arguing as in Step 1 in proof of Lemma 2 of ref. [2], we have

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1, \quad \text{for each } z \in E_R^*. \tag{3.3}$$

For each $n \in \mathbb{N}$ we put

$$Q_n(z) = a_n z^{\alpha_n} \prod_{|z_{n,j}|\leq R_n} (1 - z/z_{n,j}).$$

By (3.1) we have

$$\limsup_{n\to\infty} |Q_n(z)|^{1/k_n} = \limsup_{n\to\infty} |P_n(z)|^{1/k_n},$$

for all $z \in \mathbb{C}$. Indeed, putting $P_n(z) = Q_n(z)H_n(z)$, to prove this assertion we need only to prove that

$$\lim_{n\to\infty} |H_n(z)|^{1/k_n},$$

for any fixed $z \in \mathbb{C}$ where

$$H_n(z) = \prod_{|z_{n,j}|>R_n} (1 - z/z_{n,j}).$$

Arguing as in the proof of Lemma 3, using (3.1) we get that

$$\limsup_{n\to\infty} |H_n(z)|^{1/k_n} \leq 1.$$

By the Taylor's expansion for the function $\log(1+\epsilon)$ for $|\epsilon|$ small enough, we see that $t \geq \exp\{t - 1 - 2(1-t)^2\}$ for t near 1. Since z is fixed, for n large enough we have that $|1 - z/z_{n,j}|$ is near 1 if $z_{n,j} \geq R_n$, so using the same argument in the proof of Lemma 3 we have

$$\begin{aligned}|1 - z/z_{n,j}| &\geq \exp\left\{|1 - z/z_{n,j}| - 1 - 2(1 - |1 - z/z_{n,j}|)^2\right\}\\ &\geq \exp\left\{\frac{-z/z_{n,j} - \overline{z}/\overline{z}_{n,j} + |z|^2/|z_{n,j}|^2}{|1 - z/z_{n,j}| + 1} - 2|z|^2/|z_{n,j}|^2\right\}.\end{aligned}$$

Thus, similarly we have

$$\liminf_{n\to\infty} |H_n(z)|^{1/k_n} \geq 1.$$

Combining above results we get that

$$\lim_{n\to\infty} |H_n(z)|^{1/k_n} = 1.$$

By Lebesgue's dominated convergence theorem we get

$$\limsup_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\log|Q_n(se^{it})|^{1/k_n}\mathrm{d}t=\limsup_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\log|P_n(se^{it})|^{1/k_n}\mathrm{d}t,$$

for all $s>0$.

From (3.3) we have

$$\limsup_{n\to\infty}|Q_n(z)|^{1/k_n}\leq 1,\quad \text{for all } z\in E_R^*.$$

Applying Bernstein's inequality (see, e.g. [2]) to polynomials $Q_n's$ and arguing as in Step 2 in the proof of Lemma 2 of ref. [2] we get

$$\limsup_{n\to\infty}\log|Q_n(z)|^{1/k_n}\leq \kappa g(E_R^*,z),\quad \text{for all } z\in C,$$

where

$$\kappa=\limsup_{n\to\infty}\frac{\eta(P_n,R_n)}{k_n}<\infty.$$

Fixing $s>0$, integrating above inequality on the circle $|z|=s$, applying Fatou's Lemma (see, e.g. Lemma 1.28 in [8]), and letting $R\to\infty$ we get

$$\limsup_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\log|Q_n(se^{it})|^{1/k_n}\mathrm{d}t\leq 0.$$

Thus

$$\limsup_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\log|P_n(se^{it})|^{1/k_n}\mathrm{d}t\leq 0,$$

for all $s>0$, which gives

$$\limsup_{n\to\infty}\|P_n\|_R^{1/k_n}\leq 1,$$

for all $R>0$, by Lemma 3. ∎

Theorem 2 *Let E be a closed set such that*

$$\limsup_{R\to\infty}\frac{\log cap(E_R)}{\log R}=\beta>0. \tag{3.4}$$

Assume that (3.1) *and* (3.2) *holds. If for all $z\in E$ we have*

$$\limsup_{n\to\infty}|P_n(z)|^{1/k_n}\leq h(|z|),$$

where

$$\limsup_{R\to\infty}\frac{\log h(R)}{\log R}\leq\gamma<\infty.$$

Then for all $R > 0$ *we have*

$$\limsup_{n\to\infty} \|P_n\|_R^{1/k_n} \leq C_0(1+R)^{\gamma/\beta}.$$

Proof Take $s_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{\log cap(E_{s_n})}{\log s_n} = \beta.$$

We can assume that $E \cap \{z : |z| = s_n\} = \emptyset$ for all $n \in \mathbb{N}$. Indeed, we can choose $s_n{}'s$ such that

$$\lim_{n\to\infty} \frac{\log s_n}{\log s_{n+1}} = \lim_{n\to\infty} \frac{\log n}{\log s_n} = 0.$$

Let $E^* = E \setminus \bigcup_{n=1}^{\infty} \{z : s_n < |z| < 1 + s_n\}$. Then E^* is closed, and for all $n \in \mathbb{N}$ we have

$$cap(E^*_{s_n}) \geq cap(E_{s_n}) - cap(E_{s_{n-1}}) \geq cap(E_{s_n}) - \log s_{n-1},$$

hence

$$\lim_{n\to\infty} \frac{\log cap(E^*_{s_n})}{\log s_n} = \beta.$$

Replacing E and s_n by E^* and $s_n + (1/2)$, we see that the conditions of Theorem still holds and $E \cap \{z : |z| = s_n\} = \emptyset$ for all $n \in \mathbb{N}$. It follows that (see, e.g. formula (8.3) p. 114 in [7])

$$g(E_{s_n}, s_n e^{it}) = -\log cap(E_{s_n}) + \int_{\partial E_{s_n}} \log |s_n e^{it} - \zeta| \mathrm{d}\mu(E_{s_n}, \zeta),$$

where $n \in \mathbb{N}$, $t \in [0, 2\pi]$ and $\mu(\cdot,\cdot)$ is the harmonic measure. Integrating this identity and applying Fubini's Theorem (see, e.g. Theorem 7.8 in [8]) we get

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} g(E_{s_n}, s_n e^{it})\,\mathrm{d}t &= -\log cap(E_{s_n}) + \frac{1}{2\pi}\int_0^{2\pi}\int_{\partial E_{s_n}} \log |s_n e^{it} - \zeta| \mathrm{d}\mu(E_{s_n}, \zeta) \\
&= -\log cap(E_{s_n}) + \int_{\partial E_{s_n}} \mathrm{d}\mu(E_{s_n}, \zeta) \frac{1}{2\pi}\int_0^{2\pi} \log |s_n e^{it} - \zeta| \mathrm{d}t \\
&= -\log cap(E_{s_n}) + \int_{\partial E_{s_n}} \mathrm{d}\mu(E_{s_n}, \zeta) \log s_n \\
&= \log s_n - \log cap(E_{s_n}),
\end{aligned}$$

where we have used

$$\frac{1}{2\pi}\int_0^{2\pi} \log |s_n e^{it} - \zeta| \mathrm{d}t = \max\{\log s_n, \log |\zeta|\} = \log s_n, \qquad \int_{\partial E_{s_n}} \mathrm{d}\mu(E_{s_n}, \zeta) = 1.$$

Hence we have

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \frac{g(E^*_{s_n}, s_n e^{it})}{\log s_n} \mathrm{d}t = 1 - \beta.$$

Put, as in Theorem 1,

$$\kappa = \limsup_{n\to\infty} \frac{\eta(P_n, R_n)}{k_n}.$$

Arguing as in the proof of Theorem 1 we get

$$\limsup_{n\to\infty} \log |Q_n(z)|^{1/k_n} \le \log h(R) + \kappa g(E_R, z),$$

for all $z \in \mathbb{C}$ and $R > 0$ such that $cap(E_R) > 0$.

It follows that in view of the definition of $C(P, R)$

$$\limsup_{n\to\infty} \log C(P_n, s_m)^{1/k_n} / \log s_m \le \log h(s_m)/\log s_m + \kappa \frac{1}{2\pi} \int_0^{2\pi} \frac{g(E^*_{s_m}, s_m e^{it})}{\log s_m},$$

for all $m \in \mathbb{N}$. Hence

$$\liminf_{s\to\infty} \limsup_{n\to\infty} \log C_n(P_n, s)^{1/k_n} / \log s \le \limsup_{m\to\infty} \left[\log h(s_m)/\log s_m + \kappa \frac{1}{2\pi} \int_0^{2\pi} \frac{g(E^*_{s_m}, s_m e^{it})}{\log s_m} \right]$$
$$\le \gamma + \kappa(1 - \beta).$$

By Lemma 2 we have

$$\liminf_{s\to\infty} \limsup_{n\to\infty} \log C_n(P_n, s)^{1/k_n} / \log s \ge \kappa.$$

Thus

$$\kappa \le \gamma + \kappa(1 - \beta),$$

or

$$\kappa \le \gamma/\beta.$$

Hence

$$\liminf_{s\to\infty} \limsup_{n\to\infty} \frac{\log C(P_n, s)^{1/k_n}}{\log s} \le \gamma + \frac{\gamma}{\beta}(1 - \beta) = \frac{\gamma}{\beta}.$$

Applying Lemma 3 we get the conclusion of Theorem 2. ∎

THEOREM 3 *Assume that* $C_0^* > 0$, *and*

$$\limsup_{n\to\infty} C(P_n, R)^{1/k_n} \le 1 \tag{3.5}$$

for all $R > 0$. *We do not assume any more conditions, in particular, we do not assume that* $k_n \ge d^*(P_n)$.

Then we have the following two alternatives: Either

(1) *For any* $m \in \mathbb{N}$

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{1}{k_n} \sum_{|z_{n,j}| \ge R} \frac{1}{|z_{n,j}|^m} = \infty, \tag{3.6}$$

or,

(2) *There exists* $m \in \mathbb{N}$ *such that*

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{1}{k_n} \sum_{|z_{n,j}|\geq R} \frac{1}{|z_{n,j}|^m} = 0. \tag{3.7}$$

Moreover, if (3.7) *is the case, for any* $l \in \mathbb{N}$

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{1}{k_n} \left| \sum_{|z_{n,j}|\geq R} \frac{1}{z^l_{n,j}} \right| = 0. \tag{3.8}$$

In particular, if $k_n \geq d^*(P_n)$ *for all* $n \in \mathbb{N}$ *then alternative* 2 *holds, hence conditions* (3.1) *and* (3.2) *are satisfied.*

Proof Assume that alternative 1 is not true. Then there exists $m \in \mathbb{N}$ such that

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{1}{k_n} \sum_{|z_{n,j}|\geq R} \frac{1}{|z_{n,j}|^m} = 0. \tag{3.9}$$

From (3.5), using Lemma 2 (ii) we can choose a sequence $R_n \to \infty$ such that

$$\limsup_{n\to\infty} \frac{\eta(P_n, R_n)}{k_n} = \kappa < \infty.$$

(In fact, we can take $\kappa = 0$ under assumptions of Theorem 3. However, here we present the proof in such a way that it can be directly generated to more general cases, see the remarks right after this proof.)

Then define

$$Q_n(z) = a_n z^{\alpha_n} \prod_{|z_{n,j}|\leq R_n} (1 - z/z_{n,j}),$$

and

$$H_n(z) = \prod_{|z_{n,j}|> R_n} (1 - z/z_{n,j}).$$

Then we have

$$\limsup_{n\to\infty} C(Q_n, R)^{1/k_n} = \limsup_{n\to\infty} C(P_n, R)^{1/k_n} \leq 1$$

for all $R > 0$. Then from Lemma 3 we get

$$u(z) = \limsup_{n\to\infty} \frac{1}{k_n} \log |Q_n(z)| \leq \log C_0$$

for all $z \in \mathbb{C}$. Since $C_0^* \geq 0$, for any $\theta \in [0, 2\pi]$ and for any $s > 0$, since the set $\{\mathrm{R}e^{i\theta}\colon R \geq s\}$ is non-thin at ∞, by Proposition 1 (or Theorem 1) there exists $R > s$ such that

$$\limsup_{n\to\infty} |Q_n(\mathrm{R}e^{i\theta})|^{1/k_n} \geq C_0^*/2. \tag{3.10}$$

Use the following Taylor expansion (see also Lemma 1)

$$\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots - \frac{z^{m-1}}{m} + O(|z|^m)$$

for $|z| \leq 1/2$, by definition of $H_n(z)$ and 3.7 we have

$$\liminf_{n\to\infty} |H_n(z)|^{1/k_n}$$
$$= \liminf_{n\to\infty}\left|\exp\left\{-\frac{1}{k_n}\left[\frac{z}{1}\sum_{|z_{n,j}|\geq R}\frac{1}{z_{n,j}} + \frac{z^2}{2}\sum_{|z_{n,j}|\geq R}\frac{1}{z_{n,j}^2} + \cdots + \frac{z^{m-1}}{m-1}\sum_{|z_{n,j}|\geq R}\frac{1}{z_{n,j}^{m-1}}\right]\right\}\right|.$$

Define

$$\beta_{n,l} = -\frac{1}{k_n}\sum_{|z_{n,j}|\geq R_n}\frac{1}{l z_{n,j}^l} = |\beta_{n,l}|e^{i\theta_{n,l}}$$

for $l=1,2,\ldots$ where $\theta_{n,l}\in[0,2\pi]$ is the argument of $\beta_{n.l}$. To prove (3.8) it suffices to show that

$$\lim_{n\to\infty}|\beta_{n,l}| = 0$$

for all $l=1,2,\ldots,m-1$ (for $l\geq m$ this claim is true in view of (3.7)). Assume by contradiction that the above claim is not true. Then passing to a subsequence we may assume that

$$\lim_{n\to\infty}|\beta_{n,l}| = \beta_l,$$
$$\lim_{n\to\infty}|\theta_{n,l}| = \theta_l$$

for all $l=1,2,\ldots,m$ where we allow β_l to take the value of ∞, and at least one of β_l is positive (we include here $\beta_m=0$ for convenience). Then passing to a further subsequence we may choose an $l\in\{1,2,\ldots,m-1\}$ such that

$$\lim_{n\to\infty}\frac{|\beta_{n,l}|}{|\beta_{n,h}|} = +\infty$$

for $h > l$, and

$$\lim_{n\to\infty}\frac{|\beta_{n,l}|}{|\beta_{n,h}|} > 0$$

for $h < l$. Fix $\theta = -\theta_l/l$, choose $R > 0$ such that (3.10) is satisfied. For that R, by choosing l, we have

$$\log\frac{2C_0}{C_0^*} \geq \liminf_{n\to\infty}\left[\mathrm{Re}^{\,i\theta}\beta_{n,1} + R^2e^{2i\theta}\beta_{n,2} + \cdots + R^{m-1}e^{(m-1)i\theta}\beta_{n,m-1}\right]$$
$$\geq |R|^l|\beta_l| - |R|^{l-1}O(|\beta_l|).$$

Since R can be chosen as large as we like we conclude that $\beta_1=0$, which is a contradiction. This completes the proof of Theorem 3. ∎

The conditions (3.7) and (3.8) in the alternative (2) of Theorem 3 turns out to be enough to achieve the conclusions of Theorems 1 and 2. So in some sense it is optimal.

THEOREM 4 *If in the statements of Theorems* 1 *and* 2 *we replace the conditions* $k_n \geq d^*(P_n)$ *and* (3.1) *by the conditions* (3.7) *and* (3.8) *in the alternative* (2) *of Theorem* 3 *while keeping the other conditions, then their conclusions are still true.*

Proof The proof is the same as the proofs of Theorems 1 and 2, using the Taylor's expansion of $\log(1-z)$ as in the proof of Theorem 3. ■

Remarks

(1) The results in this section can be generalized to any sequence (P_n) of finite orders $\rho_n \leq \rho < \infty$ for all n (see the discussions in the last section).

(2) The proofs of Theorem 3 can be appropriately changed to get similar results for the case we have the more growth

$$\limsup_{n\to\infty} \|P_n\|_R^{1/k_n} \leq 1 + R$$

for all $R > 0$. Here we recall that

$$\|P_n\|_R = \sup_{|z|\leq R} |P_n(z)|.$$

(3) In the alternative 2 of Theorem 3 if we take into account the fact that we can choose $R_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{\eta(P_n, R_n)}{k_n} = 0,$$

we can write (3.8) as

$$\lim_{n\to\infty} \frac{1}{k_n} \sum_{|z_{n,j}|\geq 1} \frac{1}{z_{n,j}^l} = 0.$$

(4) The LHS of (3.8) can be effectively computed in practice. In fact, if

$$P(z) = a\prod_j \left(1 - \frac{z}{z_j}\right)$$

then formally

$$\log P(z) = \log a - \frac{z}{1}\sum_j \frac{1}{z_j} - \frac{z^2}{2}\sum_j \frac{1}{z_j^2} - \frac{z^3}{3}\sum_j \frac{1}{z_j^3} - \cdots$$

Hence we can compute

$$\sum_j \frac{1}{z_j} = -\frac{P'(0)}{P(0)}$$

and so on.

4. Some consequences and refinements

Our first example of applications in this section is the following extension of Theorem 1 in [2] (for the convenience of the reader, we state the result in a similar manner to that of Theorem 1 in [2]):

THEOREM 5 *Let* Γ *be a continuum in* $\mathbb{C}$ (*for example, a continuous curve which is not a point*), *and let* $E \subset \mathbb{C}$ *be closed such that* E *is non-thin at* ∞. *Assume that* (P_n) *is a sequence of entire functions of genus zero such that* $d^*(P_n) \leq k_n$, *where* (k_n) *is an increasing sequence of integer numbers such that conditions* (3.1) *and* (3.2) *are satisfied. Moreover, suppose that the following two conditions are satisfied*:

(i) *There is some function* f *which is analytic on some simply connected open set* $G_f \subset \mathbb{C}$ *containing* Γ *with*

$$\limsup_{n\to\infty} \|f - P_n\|_{\Gamma}^{1/k_n} < 1.$$

(ii) *For all* $z \in E$

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1.$$

Then the following statements are true:

(1) *If* (k_{n+1}/k_n) *is bounded, then* f *extends to an entire function, and for any compact set* K *of* $\mathbb{C}$, *we have*

$$\limsup_{n\to\infty} \|f - P_n\|_{K}^{1/k_n} < 1.$$

(2) *If* (k_n) *is arbitrary, then for any compact* $K \subset G_f$ *we have*

$$\limsup_{n\to\infty} \|f - P_n\|_{K}^{1/k_n} < 1.$$

Remark This theorem is an extension of Theorem 1 in [2]: in case (P_n) is a sequence of polynomials and $k_n \geq \deg(P_n)$, our result here is exactly that of Theorem 1 in [2]. In fact, if we are in case (1) that as was shown in [2], using Bernstein theorem we have that the condition (i) in our Theorem is the same as the condition (i) in Theorem [2]. If we are in case (2), then their condition and our condition are also the same.

Proof From the assumption, by Theorem 1 we have

$$\limsup_{n\to\infty} \|P_n\|_{K}^{1/k_n} \leq 1$$

for any compact set $K \subset \mathbb{C}$.

For proof of (2), we use Theorem 10 in [6].

Now we prove (1). Again, by Theorem 10 in [6], and the assumption that (k_{n+1}/k_n) is bounded, there is a non-empty open set U contained in G_f for which

$$\limsup_{n\to\infty} \|P_{n+1} - P_n\|_{U}^{1/k_n} < 1.$$

Then using the property given at the beginning of this proof, argue similarly to the proof of Theorem 1 in [2] (that is, use two-constants theorem) we get that

$$\limsup_{n\to\infty} \|P_{n+1} - P_n\|_K^{1/k_n} < 1$$

for any compact set $K \subset \mathbb{C}$. From this, it is straightforward to imply that f extends to an entire function and

$$\limsup_{n\to\infty} \|f - P_n\|_K^{1/k_n} < 1$$

for any compact set $K \subset \mathbb{C}$. ∎

Now we come to another interesting corollary, which gives a criterion for local convergence of a summation of a sequence of entire functions of genus zero.

THEOREM 6 *Assume that* (P_n) *is a sequence of entire functions of genus zero,* (k_n) *is an increasing sequence of integer numbers, and* E *is a closed subset of* $\mathbb{C}$ *such that conditions of Theorem* 2 *are satisfied. Moreover, assume that there exists* $z_0 \in \mathbb{C}$ *and* $\rho > 0$ *such that*

$$C := \limsup_{n\to\infty} \exp\left\{\frac{1}{2\pi k_n}\int_0^{2\pi} \log|P_n(z_0 + \rho\theta)|\,\mathrm{d}\theta\right\} < 1.$$

Let $\beta > 0$, γ *denote the constants in the statement of Theorem* 2. *Then* $\sum_{n=1}^{\infty} P_n$ *converges uniformly in the set* $\{|z - z_0| < \rho((1/C^{\beta/\gamma}) - 1)\}$. *In particular, if* $\gamma = 0$ *then* $\sum_{n=1}^{\infty} P_n$ *converges uniformly in* $\mathbb{C}$.

Proof Define $Q_n(z) = P_n(z_0 + \rho z)$. Because we will apply Theorem 2 for Q_n, we check here whether the conditions of Theorem 2 are satisfied for sequences (Q_n) and (k_n). First, note that the condition $k_n \geq d^*(Q_n)$ is in fact not needed in the proof of all previous results, we need only that

$$\limsup_{n\to\infty} \frac{d^*(Q_n)}{k_n} < \infty,$$

and this condition is obviously satisfied from our assumptions on P_n. We can also choose $R_n's$ for Q such that the other conditions of Theorem 2 are satisfied.

From the above analysis, without loss of generality, we may assume that $z_0 = 0$, $\rho = 1$. Then $C = C_0$, where C_0 is the constant defined in the beginning of Section 2. By Theorem 2, we have

$$\limsup_{n\to\infty} \|P_n\|_R^{1/k_n} \leq C_0(1+R)^{\gamma/\beta}$$

for any $R > 0$. Then the conclusions of Theorem 6 is straightforward to obtain. ∎

Theorem 6 can be applied to such sequences as $P_n(z) = z^n$ and $k_n = n$. In fact, the conditions of Theorem 6 are natural: If a sequence of polynomials $P_n(z)$ is bounded in any non-empty open set of $\mathbb{C}$ then

$$|P_n(z)|^{1/\deg(P_n)} \leq C(1 + |z|)$$

for all $z \in \mathbb{C}$ where C is a constant.

We finish this section by outlining some refinements (see also Theorems 3 and 4).

First, in theory we can effectively choose a sequence (k_n) satisfying condition (3.2), given a sequence of entire function of genus zero (P_n). To this end, we note that if $P(z)$ is an entire function of genus zero, then Lemma 2 in [1] shows that

$$\rho_1(P) := \limsup_{r\to\infty}\frac{\log\eta(P,r)}{\log r}\leq 1.$$

Here $\rho_1(P)$ is called the order of counting function for $P(z)$. (It is known, see Theorem 2, p. 18 in [1], that $\rho_1(P)$ does not exceed the growth order $\rho(P)$ that we defined in Section 2.) Hence, condition (3.2) is satisfied if we choose k_n as

$$k_n \approx R_n^{\rho(P_n)} \lesssim R_n.$$

Lastly, condition (3.1) is trivially satisfied if the functions $P_n(z)$ have some symmetric properties. For example, this is the case if $P_n(z)$ has the form

$$P_n(z) = \prod_m P_{n,m}(z)$$

where $P_{n,m}$ are polynomials whose sums of the reverses of zeros (roots) are zero identically, and such that the absolute value of any zero of $P_{n,m+1}$ is greater than the absolute value of any zero of $P_{n,m}$, for any m. It was observed in [1, p. 32] that this symmetric property affects the growth of an entire function.

5. Examples

The same ideas of the previous sections can be applied to various problems, some of which are described in the following.

Example 1 grouped power series.

A grouped power series which can be formally written as

$$s(z) = \sum_{n=0}^{\infty} z^{k_n} P_n(z) \tag{5.1}$$

where (k_n) is an increasing sequence of positive integers, and $P_n(z)$ are entire functions. In [2], the authors considered the case where $P_n(z)$ are polynomials of degree μ_k such that $\lambda_k + \mu_k < \lambda_{k+1}$. Here we consider a more general case that $P_n(z)$ are entire functions of genus zero.

The following result is a generalization of Corollary p. 199 in [2].

Proposition 2 *Let E be a subset of* $\mathbb{C}$ *which is non-thin at* ∞. *Let* $s(z)$ *be defined as in* (5.1) *where* (P_n) *is a sequence of entire functions of genus zero, where* (k_n) *is an increasing sequence of positive integers. Assume that* $k_n \geq d^*(P_n)$ *for all n, and conditions* (3.1) *and* (3.2) *are satisfied.*

If $s(z)$ *converges pointwise on E then it converges uniformly on compact sets of* $\mathbb{C}$.

Proof Define

$$Q_n(z) = z^{k_n} P_n(z).$$

Then the sequences (Q_n) and $(2k_n)$ satisfy conditions (3.1) and (3.2), and $2k_n \geq d^*(Q_n)$ for all n.

Now since $s(z)$ converges pointwise on E, we have

$$Q_n(z) = s_n(z) - s_{n-1}(z)$$

converges pointwise to 0 on E. In particular,

$$\limsup_{n\to\infty} |Q_n(z)|^{1/(2k_n)} \leq 1$$

for all $z \in E$. Applying Theorem 1 we get

$$\limsup_{n\to\infty} \|Q_n\|_R^{1/(2k_n)} \leq 1$$

for all $R > 0$. From the definition of Q_n we have that

$$\limsup_{n\to\infty} \|P_n\|_R^{1/(2k_n)} \leq R^{-1/2}$$

for all $R > 0$. Now using that the LHS of above inequality is not decreasing as a function of R, we have

$$\limsup_{n\to\infty} \|P_n\|_R^{1/(2k_n)} = 0$$

for all $R > 0$. Then we also have

$$\limsup_{n\to\infty} \|Q_n\|_R^{1/(2k_n)} = 0$$

for all $R > 0$. From this, it follows easily that $s(z)$ converges uniformly in every compact subset of $\mathbb{C}$. ■

Example 2 Fourier transforms with real kernels of compact support.

Let φ be a non-trivial function of $L^1(\mathbb{R})$. Then its Fourier transform is defined by

$$\widehat{\varphi}(x) = \int_{-\infty}^{\infty} e^{-ixt}\varphi(t)\,\mathrm{d}t, \quad x \in \mathbb{R}.$$

If φ is of compact support, there associates an entire function of exponential type of Laplace transform type

$$\Phi(z) = \int_{-\infty}^{\infty} e^{zt}\varphi(t)\,\mathrm{d}t. \tag{5.2}$$

From Lemma 3 in [9], if $\varphi(t) = 0$ for all $t \leq 0$ then $\Phi(z)$ has a representation

$$\Phi(z) = Ce^{z(\sigma+\mu)/2}z^{\alpha} \prod_{z_j\in\mathbb{R}}\left(1 - \frac{z}{z_j}\right) \prod_{Imz_j>0}\left(1 - \frac{z}{z_j}\right)\left(1 - \frac{z}{\overline{z_j}}\right), \tag{5.3}$$

where

$$\sigma = \sup\{a > 0 : \varphi|_{[0,a]} = 0 \ a.e.\},$$
$$\mu = \inf\{a \in [0,\infty) : \varphi|_{[a,\infty)} = 0 \ a.e.\}.$$

Moreover

$$\sum_j \left|\mathrm{Re}\left(\frac{1}{z_j}\right)\right| < \infty,$$
$$\sum_j \frac{1}{|z_j|^q} < \infty$$

for all $q > 1$. From (5.3) we can explore the growth of a sequence of entire functions (Φ_n) which are Laplace transforms of real kernel with compact support in a manner that is similar to that of Sections 2–4, under appropriate changes on the assumptions (See Theorem 7 for one way to doing that). For example, we may define the degree of an entire function $\Phi(z)$ having the representation (5.3) by

$$d^*(\Phi) = \sigma + \mu + \alpha + \sum_j \left|\mathrm{Re}\,\frac{1}{z_j}\right| + \sum_j \frac{1}{|z_j|^q}$$

for some fixed $1 < q \leq 2$. However, we will not pursue this issue in this section (see also discussions in the last section).

Now we go back to the Laplace transform (5.2). Generally, if $\varphi \in L^1(\mathbb{R})$ satisfies

$$\int_{-\infty}^{\infty} e^{|t|s}|\varphi(t)|\,\mathrm{d}t < \infty, \tag{5.4}$$

for all $s > 0$ then Φ in (5.2) is defined on all the complex plane $\mathbb{C}$ and is an entire function, but its order may be any positive number. If $\varphi \geq 0$ then condition (5.4) is also necessary for Φ to be an entire function. Indeed, in this case we have, for every $R > 0$,

$$\max_{|z|\leq R} |\Phi(z)| \geq \frac{1}{2}(\Phi(R) + \Phi(-R)) \geq \frac{1}{2}\int_{-\infty}^{\infty} e^{|t|R}\varphi(t)\,\mathrm{d}t.$$

Condition (5.4) is rather strict. We may ask a more general question: For what function φ being non-trivial and in $L^1_{loc}(\mathbb{R})$ (meaning $\varphi \in L^1(K)$ for all compact set K of $\mathbb{R}$) that the sequence

$$\Phi_n(z) = \int_{-n}^{n} e^{zt}\varphi(t)\,\mathrm{d}t \tag{5.5}$$

converges locally uniformly in $\mathbb{C}$ to an entire non-zero function $\Phi(z)$? Applying the results in previous sections, we will show that there is an obstruction to this question, which can be presented in terms of zeros of $\Phi_n(z)$ (Theorem 7).

For simplicity we will assume that $\varphi(t) = 0$ for $t \leq 0$. From formula (5.3), for n large enough, Φ_n has a representation

$$\Phi_n(z) = C_n e^{z(\sigma+\mu_n)/2} z^{\alpha_n} \prod_{z_{j,n}\in\mathbb{R}} \left(1 - \frac{z}{z_{j,n}}\right) \prod_{\mathrm{Im}\, z_{j,n}>0} \left(1 - \frac{z}{z_{j,n}}\right)\left(1 - \frac{z}{\overline{z_{j,n}}}\right), \tag{5.6}$$

where

$$\sigma = \sup\{a > 0 : \varphi|_{[0,\,a]} = 0 \ a.e.\},$$
$$\mu_n = \inf\{a \in [0, n] : \varphi|_{[a,\,n]} = 0 \ a.e.\}.$$

Moreover

$$\sum_j \left|\mathrm{Re}\left(\frac{1}{z_{j,n}}\right)\right| < \infty,$$
$$\sum_j \frac{1}{|z_{j,n}|^q} < \infty$$

for all $q > 1$.

Then we have the following result:

THEOREM 7 *Given* $\varphi \in L^1_{loc}(\mathbb{R})$, $\varphi(t) = 0$ *for* $t \le 0$, *and the support of* φ *is non-compact. Let* Φ_n *be defined as above, and let* (5.6) *be its representation. Let* σ *and* μ_n *be defined as above. Assume that the following three conditions are true*

(1)

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{1}{\mu_n} \sum_{|z_{j,n}|\ge R} \left|\mathrm{Re}\left(\frac{1}{z_{j,n}}\right)\right| < \infty.$$

(2)

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{1}{\mu_n} \left|\sum_{|z_{j,n}|\ge R} \frac{1}{z_{j,n}}\right| = 0.$$

(3) *There exists* $1 < q < 2$ *such that*

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{1}{\mu_n} \sum_{|z_{j,n}|\ge R} \frac{1}{|z_{j,n}|^q} < \infty.$$

Then there is no non-zero entire function $\Phi(z)$ *such that* $\Phi_n(z)$ *converges locally uniform in* $\mathbb{C}$ *to* $\Phi(z)$.

Proof We prove by contradiction. Assume by contradiction that $\Phi_n(z)$ converges locally uniform in $\mathbb{C}$ to a non-zero entire function $\Phi(z)$.

Since the support of φ is non-compact, we have

$$\lim_{n\to\infty} \mu_n = \infty. \tag{5.7}$$

Hence

$$\lim_{n\to\infty} |\Phi_n(z)|^{1/\mu_n} \le 1,$$

for all $z \in \mathbb{C}$, and the equality is true at every $z \in \mathbb{C}$ with $\Phi(z) \ne 0$. Using (5.7) and Rouche's theorem (applied for the sequence Φ_n converging to Φ) we can find a sequence $R_n \to \infty$ such that

$$\limsup_{n\to\infty} \frac{\eta(\Phi_n, R_n)}{\mu_n} \le 1.$$

Now define

$$Q_n(z) = C_n z^{\alpha_n} \prod_{|z_{j,n}| \le R_n} \left(1 - \frac{z}{z_{j,n}}\right).$$

Use conditions 1, 2, 3 of Theorem 7, arguing similar to the proofs of results in Sections 2 and 3, for every $z \in \mathbb{C}$

$$\lim_{n\to\infty} |e^{z/2}| \cdot |Q_n(z)|^{1/\mu_n} \le 1.$$

Fix $R > 0$. If $z \in \mathbb{R}$ and $z \ge 2R$ then

$$\lim_{n\to\infty} |Q_n(z)|^{1/\mu_n} \le e^{-R}.$$

Since the set $\{z \in \mathbb{R}\colon\ z \ge R\}$ is non-thin at ∞, from the above inequality we have by Theorem 1

$$\lim_{n\to\infty} |Q_n(z)|^{1/\mu_n} \le e^{-R}$$

for all $z \in \mathbb{C}$ and for all $R > 0$. Since $R > 0$ is arbitrary expression we have from the above that

$$\lim_{n\to\infty} |Q_n(z)|^{1/\mu_n} = 0$$

for all $z \in \mathbb{C}$. Then we have

$$\lim_{n\to\infty} |\Phi_n(z)|^{1/\mu_n} = 0$$

for all $z \in \mathbb{C}$, which is a contradiction. ■

Refer to Theorem 3 where we note that the LHS of (3.8) can be computed via the integrals of $\varphi(t)$. For example, we have:

PROPOSITION 3 *Let $\varphi \in L^1(\mathbb{R})$ of compact support and $\varphi(t) = 0$ for all $t \le 0$. Define $\Phi(z)$ its Laplace transform by formula* (5.2) *with representation* (5.3). *Assume that* $\Phi(0) \ne 0$. *Then*

$$\sum_j \frac{1}{z_j} = \frac{\sigma+\mu}{2} - \frac{\Phi'(0)}{\Phi(0)} = \frac{\sigma+\mu}{2} - \frac{\int_{-\infty}^{\infty} t\varphi(t)\,\mathrm{d}t}{\int_{-\infty}^{\infty} \varphi(t)\,\mathrm{d}t} \tag{5.8}$$

Proof Formally we have

$$\log\Phi(z) = \log C + \frac{\sigma+\mu}{2} z + \sum_{z_j\in\mathbb{R}} \log\left(1 - \frac{z}{z_j}\right) + \sum_{Im z_j > 0} \log\left[1 - z\left(\frac{1}{z_j} + \frac{1}{\overline{z_j}}\right) + \frac{z^2}{|z_j|^2}\right]. \tag{5.9}$$

Then compute

$$\frac{\mathrm{d}}{\mathrm{d}z} \log\Phi(z)|_{z=0}$$

using formulas (5.2) and (5.9) we get (5.8). ■

Example 3 Grouped Fourier series.
Similarly to above Examples, we can consider series of the form

$$s(z) = \sum_j z^{k_n} P_n(z) \sin(\lambda_n z)$$

where $k_n \geq n$, P_n is a polynomial of degree $\deg(P_n) \leq k_n$, and $\lambda_n \in \mathbb{C}$ which satisfies

$$\lim_{n\to\infty} \frac{\lambda_n}{k_n} = 0.$$

Claim *If* $P_n(-z) = -P_n(z)$ *for all* $z \in \mathbb{C}$ and $n \in \mathbb{N}$ (or $P_n(-z) = P_n(z)$ *for all* $z \in \mathbb{C}$ *and* $n \in \mathbb{N}$), *and* $s(z)$ *converges on a set E such that the set* $\{z^2 \colon z \in E\}$ *is non-thin at* ∞, *then* $s(z)$ *converges locally uniformly on* $\mathbb{C}$.

This claim can be proved in the same spirit as that of Proposition 2, using the factorization of the function $\sin z$.

6. Conclusions

6.1. *The growth of sequences of bounded orders*

We can develop the parallelism of results in Sections 2–5 to this general class of entire functions. We will not go into details but will outline here how we can do for the general case.

(i) Theorem 3 can be proved in this generality with only a little change in the conclusion.
(ii) Now we show how we can generalize Theorem 1.

We will assume that (P_n) is a sequence of entire functions of finite orders $\rho_n \leq \rho$ where $\rho > 0$ is a constant, with representation

$$P_n(z) = a_n z^{m_n} e^{W_{q_n}(z)} \prod_j G\left(\frac{z}{z_{n,j}}, p_n\right) \tag{6.1}$$

where $p_n = [\rho_n]$ the integer part of ρ_n, $W_{q_n}(z)$ is a polynomial of degree $q_n \leq \rho_n$ with $W_{q_n}(0) = 0$, $z_{n,j}'s$ are zeros of $P(z)$ different from 0. Let us assume that (k_n) is a sequence of positive numbers with $k_n \geq d^*(P_n)$ for all $n \in \mathbb{N}$. (A less strong condition on k_n can also be used (Theorem 4), however the presentation of the assumptions will be very complicated, so we will not pursue this here.)

We also assume the condition (3.2), while condition (3.1) is replaced by

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{1}{k_n} \left| \sum_{|z_{n,j}|>R} \frac{1}{z_{n,j}^{p_n+1}} \right| = 0. \tag{6.2}$$

Under these conditions we have

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} = \limsup_{n\to\infty} |Q_n(z)|^{1/k_n}$$

for all $z \in \mathbb{C}$, where

$$Q_n(z) = \exp\left\{ W_{q_n}(z) + \sum_{|z_{n,j}| \le R_n} \left(\frac{z}{z_{n,j}} + \frac{z^2}{2z_{n,j}^2} + \cdots + \frac{z^{p_n}}{p_n z_{n,j}^{p_n}} \right) \right\} \times a_n z^{m_n} \prod_{|z_{n,j}| \le R_n} \left(1 - \frac{z}{z_{n_j}} \right). \quad (6.3)$$

Also, from the assumptions we have

$$u(z) = \limsup_{n\to\infty} \frac{1}{k_n} \log \left| a_n z^{m_n} \prod_{|z_{n,j}| \le R_n} \left(1 - \frac{z}{z_{n_j}} \right) \right| \le \kappa \log^+ |z| + C$$

for all $a \in \mathbb{C}$ for some constants κ, $C > 0$, and all polynomials

$$\frac{1}{k_n} \left[W_{q_n}(z) + \sum_{|z_{n,j}| \le R_n} \left(\frac{z}{z_{n,j}} + \frac{z^2}{2z_{n,j}^2} + \cdots + \frac{z^{p_n}}{p_n z_{n,j}^{p_n}} \right) \right]$$

are polynomials of degree less than $\rho + 1$, and have bounded coefficients. Passing to subsequences we may assume that

$$\lim_{n\to\infty} \frac{1}{k_n} \left[W_{q_n}(z) + \sum_{|z_{n,j}| \le R_n} \left(\frac{z}{z_{n,j}} + \frac{z^2}{2z_{n,j}^2} + \cdots + \frac{z^{p_n}}{p_n z_{n,j}^{p_n}} \right) \right] = W(z)$$

exists for all $z \in \mathbb{C}$ and is also a polynomial of degree $\deg(W) \le \rho + 1$. Then

$$\limsup_{n\to\infty} \frac{1}{k_n} \log |P_n(z)| = \mathrm{Re}\,(W(z)) + u(z). \quad (6.4)$$

From the above analysis, it is clear to see that in order to extend Theorem 1 we need the condition on the set E such that if

$$\mathrm{Re}\,(W(z)) + u(z) \le 0$$

for all $z \in E$, where $W(z)$ is a polynomial of degree $\deg(W) \le \rho + 1$, $W(0) = 0$, $\max_{|z|\le 1} |W(z)| \le 1$, and $u(z)$ is a subharmonic function with growth bounded from above by $\kappa \log^+ |z| + C$ then

$$\mathrm{Re}\,(W(z)) + u(z) \le 0$$

for all $z \in \mathbb{C}$. We do not know what is a sufficient and necessary conditions for sets E having this property (we will discuss more about this point in the second part). Here we give a sufficient condition for that property:

Proposition 4 *Let $E \subset \mathbb{C}$ be a collection of real lines L going through 0 such that for all $(1+2k)!$-th roots ζ of unity (meaning $\zeta^{(1+2k)!} = 1$) and for all line $L \subset E$, then $\zeta L = \{\zeta z : z \in L\} \subset E$ (here $k! = 1, 2, \ldots, k$ is the factorial of k). Let $W(z)$ be a polynomial of degree $\deg(W) \le k$, $W(0) = 0$. Let $u(z)$ be a subharmonic function with growth bounded from above by $\kappa \log^+ |z| + C$. If*

$$\mathrm{Re}\,(W(z)) + u(z) \le 0$$

for all $z \in E$ *then*

$$\mathrm{Re}\,(W(z)) + u(z) \leq 0$$

for all $z \in \mathbb{C}$.

Proof If $W \equiv 0$ then since E is non-thin at ∞ we are done.

Assume that $W \not\equiv 0$. Then $1 \leq \deg(W) = l \leq k$. Since the number of common zeros of the system

$$\mathrm{Re}\,(z^l) = 0,$$
$$|z| = 1$$

is not greater than $2l$, from the assumptions it is easy to see that there exists at least one line $L \subset E$ such that

$$\lim_{z \in L,\, |z| \to \infty} \mathrm{Re}\,(W(z)) = +\infty.$$

Then we even get a more stronger conclusion $u(z) \equiv -\infty$. ■

(iii) Theorem 2 can also be generalized in the same manner. We skip the details here.

6.2. *Extremal functions*

Let E be a subset of $\mathbb{C}$. The discussion in part (1) shows that it is natural to consider the following function

$$V_{E,q}(z) = \sup\{U(z) : U \in \mathbb{L}_q, U|_E \leq 0\} \tag{6.5}$$

where $\mathbb{L}_q$ is the set of all functions $U(z)$: $\mathbb{C} \to [-\infty, +\infty)$ having the form

$$U(z) = \mathrm{Re}\,(W(z)) + u(z)$$

where $W(z)$ is a polynomial of degree $\deg(W) \leq q$ and $W(0) = 0$, $\max_{|z| \leq 1} |W(z)| \leq 1$, and where $u(z) \leq \log^+|z| + C$ is a subharmonic function where C is a constant depending on u.

The function $V_{E,q}(z)$ in (6.5) is an analogous of the Siciak's extremal function [10]. Similar functions were also considered in [3]. We hope to return to this in a future article.

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