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Degree Complexity of Birational Maps Related to Matrix Inversion

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Abstract. For a $q \times q$ matrix $x = (x_{i,j})$ we let $J(x) = (x_{i,j}^{-1})$ be the Hadamard inverse, which takes the reciprocal of the elements of x. We let $I(x) = (x_{i,j})^{-1}$ denote the matrix inverse, and we define $K = I \circ J$ to be the birational map obtained from the composition of these two involutions. We consider the iterates $K^n = K \circ \cdots \circ K$ and determine degree complexity of K, which is the exponential rate of degree growth $\delta(K) = \lim_{n\to\infty} (deg(K^n))^{1/n}$ of the degrees of the iterates. Earlier studies of this map were restricted to cyclic matrices, in which case K may be represented by a simpler map. Here we show that for general matrices the value of $\delta(K)$ is equal to the value conjectured by Anglès d'Auriac, Maillard and Viallet.

§0. Introduction

Let $\mathcal{M}_q$ denote the space of $q \times q$ matrices with coefficients in $\mathbf{C}$, and let $\mathbf{P}(\mathcal{M}_q)$ denote its projectivization. We consider two involutions on the space of matrices: $J(x) = (x_{i,j}^{-1})$ takes the reciprocal of each entry of the matrix $x = (x_{i,j})$, and $I(x) = (x_{i,j})^{-1}$ denotes the matrix inverse. The composition $K = I \circ J$ defines a birational map of $\mathbf{P}(\mathcal{M}_q)$.

For a rational self-map f of projective space, we may define its nth iterate $f^n = f \circ \cdots \circ f$, as well as the degree $deg(f^n)$. The degree complexity or dynamical degree is defined as

$$\delta(f) := \lim_{n\to\infty} (deg(f^n))^{1/n}.$$

In general it is not easy to determine $\delta(f)$, or even to make a good numerical estimate. Birational maps in dimension 2 were studied in [DF], where a technique was given that, in principle, can be used to determine $\delta(f)$. This method, however, does not carry over to higher dimension. In the case of the map K_q, the dimension of the space and the degree of the map both grow quadratically in q, so it is difficult to write even a small composition $K_q \circ \cdots \circ K_q$ explicitly. This paper is devoted to determining $\delta(K_q)$.

Theorem. *If $q \geq 5$, then $\delta(K_q) > 1$ is the largest root of the polynomial $\lambda^2 - (q^2 - 4q + 2)\lambda + 1$.*

The map K and the question of determining its dynamical degree have received attention because K may be interpreted as acting on the space of matrices of Boltzmann weights and as such represents a basic symmetry in certain problems of lattice statistical mechanics (see [BHM], [BM]). In fact there are many K-invariant subspaces $T \subset \mathbf{P}(\mathcal{M}_q)$ (see, for instance, [AMV1] and [PAM]), and it is of interest to know the values of the restrictions $\delta(K|_T)$. The first invariant subspaces that were considered are $\mathcal{S}_q$, the space of symmetric matrices, and $\mathcal{C}_q$, the cyclic (also called circulant) matrices. The value $\delta(K|_{\mathcal{C}_q})$ was found in [BV], and another proof of this was given in [BK1]. Anglès d'Auriac, Maillard and Viallet [AMV2] developed numerical approaches to finding δ and found approximate values of $\delta(K_q)$ and $\delta(K|_{\mathcal{S}_q})$ for $q \le 14$. A comparison of these values with the (known) values of $\delta(K|_{\mathcal{C}_q})$ led them to conjecture that $\delta(K|_{\mathcal{C}_q}) = \delta(K_q) = \delta(K|_{\mathcal{S}_q})$ for all q.

The Theorem above proves the first of these conjectured equalities. We note that the second equality, $\delta(K|_{\mathcal{S}_q}) = \delta(K_q)$, has been recently established in [T]. This involves additional symmetry, which adds another layer of subtlety to the problem. An example where additional symmetry leads to additional complication was seen already with the K-invariant space $\mathcal{C}_q \cap \mathcal{S}_q$: the value of $\delta(K_{\mathcal{C}_q \cap \mathcal{S}_q})$ has been determined in [AMV2] (for prime q) and [BK2] (for general q), and in the general case it depends on q in a rather involved way. The reason why the cyclic matrices were handled first was that $K|_{\mathcal{C}_q}$ (see [BV]) and $K|_{\mathcal{C}_q \cap \mathcal{S}_q}$ (see [AMV2]) can be converted to maps of the form $L \circ J$ for certain linear L. In the case of $K|_{\mathcal{C}_q}$, the associated map is "elementary" in the terminology of [BK1], whereas $K|_{\mathcal{C}_q \cap \mathcal{S}_q}$ exhibits more complicated singularities, i.e., blow-down/blow-up behavior. In contrast, the present paper treats matrices in their general form, so our methods should be applicable to wider classes of K-invariant subspaces.

The degree of K is the degree of K^*H, the pullback of a (general) linear hypersurface H. A difficulty is that frequently $(K^*)^n \ne (K^n)^*$ on $H^{1,1}$. To deal with this we analyze the blow-down behavior of K, which means that we look at the hypersurfaces E for which $K(E)$ has codimension ≥ 2. We will construct a new manifold $\pi : \mathcal{X} \to \mathbf{P}(\mathcal{M}_q)$ by blowing up certain of the sets $K(E)$. The map π is a birational equivalence which changes the topology and increases the size of $H^{1,1}$. The birational map $K_{\mathcal{X}} := \pi^{-1} \circ K \circ \pi$ of $\mathcal{X}$ induces a well-defined pullback on $H^{1,1}(\mathcal{X})$. The exponential growth rate of degree is equal to the exponential growth rate of the induced maps on cohomology:

$$\delta(K) = \lim_{n\to\infty} \left(||(K^n_{\mathcal{X}})^*||_{H^{1,1}(\mathcal{X})} \right)^{1/n}.$$

Our approach is to choose a space $\mathcal{X}$ for which we can determine $(K^n_{\mathcal{X}})^*$ and thus $\delta(K)$.

In general, $deg(K \circ K) \le deg(K)^2$, so $\delta(K) \le deg(K)$. On the other hand, δ decreases when we restrict to a linear subspace, so $\delta(K) \ge \delta(K|_{\mathcal{C}_q})$. The paper [BV] shows that $\delta(K|_{\mathcal{C}_q})$ is the largest root of the polynomial $\lambda^2 - (q^2 - 4q + 2)\lambda + 1$, so it will suffice to show that this number is also an upper bound for $\delta(K)$. In order to find the right upper bound on $\delta(K_q)$, we construct a blowup space $\pi : \mathcal{Z} \to \mathbf{P}(\mathcal{M}_q)$. A basic property is that the spectral radius (or modulus of the largest eigenvalue) of the induced map $K^*_{\mathcal{Z}}$ on $H^{1,1}(\mathcal{Z})$ gives an upper bound for $\delta(K)$. Thus the goal of this paper is to construct a space $\mathcal{Z}$ such that the spectral radius of $K^*_{\mathcal{Z}}$ is the number given in the Theorem.

§1. Basic properties of I, J, and K

For $1 \le j \le q-1$, define R_j as the set of matrices in $\mathcal{M}_q$ of rank less than or equal to j. In $\mathbf{P}(\mathcal{M}_q)$, R_1 consists of matrices of rank exactly 1 since the zero matrix is not in $\mathbf{P}(\mathcal{M}_q)$. For $\lambda, \nu \in \mathbf{P}^{q-1}$, let $\lambda \otimes \nu = (\lambda_i \nu_j) \in \mathbf{P}(\mathcal{M}_q)$ denote the outer vector product. The map

$$\mathbf{P}^{q-1} \times \mathbf{P}^{q-1} \ni (\lambda, \nu) \mapsto \lambda \otimes \nu \in R_1 \subset \mathbf{P}(\mathcal{M}_q)$$

is biholomorphic, and thus R_1 is a smooth submanifold.

We let $I : \mathbf{P}(\mathcal{M}_q) \to \mathbf{P}(\mathcal{M}_q)$ denote the birational involution given by matrix inversion $I(A) = A^{-1}$. We let $x_{[k,m]}$ denote the $(q-1) \times (q-1)$ sub-matrix of $(x_{i,j})$ which is obtained by deleting the k-th row and the m-th column. We recall the classic formula $I(x) = (det(x))^{-1}\hat{I}(x)$, where $\hat{I} = (\hat{I}_{i,j})$ is the homogeneous polynomial map of degree $q-1$ given by the cofactor matrix

$$\hat{I}_{i,j}(x) = C_{j,i}(x) = (-1)^{i+j} det(x_{[j,i]}). \tag{1.1}$$

Thus $\hat{I}$ is a homogeneous polynomial map which represents I as a map on projective space. We see that $\hat{I}(x) = 0$ exactly when the determinants of all $(q-1) \times (q-1)$ minors of x vanish, i.e., when $x \in R_{q-2}$.

We may always represent a rational map $f = [f_1 : \cdots : f_{q^2}]$ of projective space $\mathbf{P}^{q^2-1}$ in terms of homogeneous polynomials of the same degree and without common factor. We define the degree of f to be the degree of f_j, and the indeterminacy locus is defined as $\mathcal{I}(f) = \{f_1 = \cdots = f_{q^2} = 0\}$. The indeterminacy locus represents the points where it is not possible to extend f, even as a continuous mapping. The indeterminacy locus always has codimension at least 2. In the case of the rational map I, the polynomials $C_{j,i}(x)$ have no common factor. Further, $\hat{I}(x) = 0$ exactly when $x \in R_{q-2}$, so it follows that the indeterminacy set is $\mathcal{I}(I) = R_{q-2}$.

We let $J : \mathbf{P}(\mathcal{M}_q) \to \mathbf{P}(\mathcal{M}_q)$ be the birational involution given by $J(x) = (J(x)_{i,j}) = (1/x_{i,j})$, which takes the reciprocal of all the entries. In the sequel, we will sometimes write $J(x) = \frac{1}{x}$. We may define

$$\hat{J}(x) = J(x)\Pi(x) \tag{1.2}$$

where $\Pi(x) = \prod x_{a,b}$ is the homogeneous polynomial of degree q^2 obtained by taking the product of all the entries $x_{a,b}$ of x, and $\hat{J}(x) = (\hat{J}_{i,j})$ is the matrix of homogeneous polynomials of degree $q^2 - 1$ such that $\hat{J}_{i,j} = \prod_{(a,b)\neq(i,j)} x_{a,b}$ is the product of all the $x_{a,b}$ except $x_{i,j}$. Thus $\hat{J}$ is the projective representation of J in terms of homogeneous polynomials.

We define $K = I \circ J$. On projective space the map K is represented by the polynomial map (1.4) below. Since $\hat{I} \circ \hat{J}$ has degree $(q-1)(q^2-1)$, we see from Proposition 1.1, that the entries of $\hat{I} \circ \hat{J}$ must have a common factor of degree $q^3 - 2q^2$.

When V is a variety, we write $K(V) = W$ for the strict transform of V under K, which is the same as the closure of $K(V - \mathcal{I}(K))$. We say that a hypersurface V is exceptional if $K(V)$ has codimension at least 2. The map I is a biholomorphic map from $\mathcal{M}_q - R_{q-1}$ to itself, so the only possible exceptional hypersurface for I is R_{q-1}. We define

$$\Sigma_{i,j} = \{x = (x_{k,\ell}) \in \mathcal{M}_q : x_{i,j} = 0\}. \tag{1.3}$$

The map J is a biholomorphic map of $\mathcal{M}_q - \bigcup_{i,j} \Sigma_{i,j}$ to itself, and the exceptional hypersurfaces are the $\Sigma_{i,j}$. Further, the indeterminacy locus is

$$\mathcal{I}(J) = \bigcup_{(a,b)\neq(c,d)} \Sigma_{a,b} \cap \Sigma_{c,d}.$$

Proposition 1.1. *The degree of K is $q^2 - q + 1$. Its representation $\hat{K} = (\hat{K}_{i,j})$ in terms of homogeneous polynomials is given by*

$$\hat{K}_{i,j}(x) = C_{j,i}\,(1/x)\,\Pi(x) \tag{1.4}$$

where $C_{j,i}$ and Π are as in (1.1) and (1.2).

Proof. Observe that $C_{j,i}(1/x)$ is independent of the variable $x_{j,i}$, while $\hat{K}(x)_{i,j}$ is not divisible by the variables $x_{k,\ell}$ with $k \neq j$ and $\ell \neq i$. Hence the greatest common divisor of all polynomials on the right hand side of (1.4) is 1. Thus the algebraic degree of K is equal to the degree of $\hat{K}(x)_{i,j}$, which is $q^2 - q + 1$. □

§2. Construction of $\mathcal{R}^1$

Divisors $V = \sum c_j V_j$ and $V' = \sum c'_k V'_k$ on a manifold $\mathcal{Z}$ are said to be linearly equivalent if there is a rational function r on $\mathcal{Z}$ so that the divisor of r is $V - V'$. The Picard group $Pic(\mathcal{Z})$ is the set of divisors modulo linear equivalence. We will construct a complex manifold $\pi : \mathcal{Z} \to \mathbf{P}(\mathcal{M}_q)$ by performing a series of blowups, and we consider the induced birational map $K_{\mathcal{Z}} := \pi^{-1} \circ K \circ \pi$. For spaces obtained by iterated blowups of $\mathbf{P}^n$, it is a standard result that the cohomology group $H^{1,1}$ is isomorphic to Pic, and in the sequel, we find it more convenient to work with Pic .

For our construction of $\mathcal{Z}$ we first blow up the spaces R_1 and $A_{i,j}$, $1 \leq i, j \leq q$. The exceptional (blowup) hypersurfaces will be denoted $\mathcal{R}^1$ and $\mathcal{A}^{i,j}$. Then we will blow up surfaces $B_{i,j} \subset \mathcal{A}^{i,j}$, which will create exceptional hypersurfaces $\mathcal{B}^{i,j}$. The precise nature of $\mathcal{Z}$ depends on the order in which the various blowups are performed. Different orders of blowup will produce different spaces $\mathcal{Z}$, but the identity map of $\mathbf{P}(\mathcal{M}_q)$ to itself induces a birational equivalence between the spaces, and this equivalence induces the identity map on $Pic(\mathcal{Z})$ ($\cong H^{1,1}(\mathcal{Z})$). Any of these spaces $\mathcal{Z}$ yields an induced birational map $K_{\mathcal{Z}}$, and each $K_{\mathcal{Z}}$ induces essentially the same pullback map $K^*_{\mathcal{Z}}$ on $Pic(\mathcal{Z})$. This issue is discussed in [BK2, §2].

We start our discussion with R_1. Let $\pi_1 : \mathcal{Z}_1 \to \mathbf{P}(\mathcal{M}_q)$ denote the blowup of $\mathbf{P}(\mathcal{M}_q)$ along R_1. We will give a coordinate chart for points of $\mathcal{Z}_1$ lying over a point $x^0 \in R_1$. Let us first make a general observation. Let $\rho_{\ell,m}$ denote the matrix operation which interchanges the ℓ-th and m-th rows of a matrix $x \in \mathcal{M}_q$, and let $\gamma_{\ell,m}$ denote the interchange of the ℓ-th and m-th columns. It is evident that J commutes with both $\rho_{\ell,m}$ and $\gamma_{\ell,m}$, whereas we have $\rho_{\ell,m}(I(x)) = I(\gamma_{\ell,m}(x))$. Thus, for the purposes of looking at the induced map $K_{\mathcal{Z}_1}$, we may permute the coordinates of $(x_{i,j})$, and without loss of generality we may assume that the (1,1) entry of x^0 does not vanish. This means that we may assume that $x^0 = \lambda^0 \otimes \nu^0$ with $\lambda^0, \nu^0 \in U_1$, where $U_1 = \{z = (z_1, \ldots, z_q) \in \mathbf{C}^q : z_1 = 1\}$.

We write the standard affine coordinate charts for $\mathbf{P}(\mathcal{M}_q)$ as

$$W_{r,s} = \{x \in \mathcal{M}_q : x_{r,s} = 1\} \subset \mathbf{C}^{q^2}, \tag{2.1}$$

where $1 \leq r, s \leq q$. Let us define V to be the set of all matrices $x \in \mathcal{M}_q$ such that the first row and column vanish. Further, for $2 \leq k, \ell \leq q$, we define a subset of V:

$$V_{k,\ell} = \{x \in \mathcal{M}_q : x = \begin{pmatrix} 0 & 0 \\ 0 & x_{[1,1]} \end{pmatrix} \text{ and } x_{k,\ell} = 1\}. \tag{2.2}$$

Now we may represent a coordinate neighborhood of $\mathcal{Z}_1$ over x^0 as

$$\pi_1 : \mathbf{C} \times U_1 \times U_1 \times V_{k,\ell} \to W_{1,1}, \quad \pi_1(s, \lambda, \nu, v) = \lambda \otimes \nu + sv. \tag{2.3}$$

Since $\lambda \otimes \nu$ has rank 1 and nonvanishing (1,1) entry, we see that $\pi_1(s, \lambda, \nu, v) \in R_1$ exactly when $s = 0$. Thus the points of $\mathcal{R}^1$ which are in this coordinate neighborhood are given by $\{s = 0\}$. If $y \in \mathcal{M}_q$ is a matrix with $y_{k,\ell} \neq 0$, then we find $\pi_1^{-1}(y) = (s, \lambda, \nu, v)$, where

$$\tilde{y} = y/y_{k,\ell}, \quad s = y_{k,\ell}, \quad \lambda = \tilde{y}_{*,1}, \quad \nu = \tilde{y}_{1,*}, \quad v = s^{-1}(\tilde{y} - \lambda \otimes \nu). \tag{2.4}$$

We may write the induced map $K_{\mathcal{Z}_1} = \pi_1^{-1} \circ K \circ \pi_1$ in a neighborhood of $\mathcal{R}^1$ by using the coordinate projections (2.3) and (2.4). This allows us to show that $K_{\mathcal{Z}_1}|_{\mathcal{R}^1}$ has a relatively simple expression:

Proposition 2.1. *We have $K_{\mathcal{Z}_1}(\mathcal{R}^1) = R_{q-1}$, so $\mathcal{R}^1$ is not exceptional for $K_{\mathcal{Z}_1}$. In fact for $z_0 = \pi_1(0, \lambda, \nu, v) \in \mathcal{R}^1$,*

$$K_{\mathcal{Z}_1}(z) = B \begin{pmatrix} 0 & 0 \\ 0 & I_{q-1}(v') \end{pmatrix} A \tag{2.5}$$

where I_{q-1} denotes matrix inversion on $\mathcal{M}_{q-1}$, and

$$v' = \left(\frac{-v_{j,k}}{\lambda_j^2 \nu_k^2} \right)_{2 \le j,k \le q}, \; A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\lambda_2^{-1} & 1 & & \\ \vdots & & \ddots & \\ -\lambda_q^{-1} & & & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & -\nu_2^{-1} & \cdots & -\nu_q^{-1} \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}. \tag{2.6}$$

Proof. Without loss of generality, we work at points $\lambda, \nu \in U_1$ such that $\lambda_j, \nu_k \neq 0$ for all j, k and V such that the v' in (2.6) is invertible. Then

$$J(\pi_1(s, \lambda, \nu, v)) = \frac{1}{\lambda \otimes \nu} + s \begin{pmatrix} 0 & 0 \\ 0 & v' \end{pmatrix} + O(s^2) = \pi_1(s + O(s^2), \lambda^{-1}, \nu^{-1}, v' + O(s)). \tag{2.7}$$

Observe that

$$A \left(\frac{1}{\lambda \otimes \nu} \right) B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$s\, A \begin{pmatrix} 0 & 0 \\ 0 & v' \end{pmatrix} B = \begin{pmatrix} 0 & 0 \\ 0 & sA_{[1,1]} v' B_{[1,1]} \end{pmatrix}.$$

Thus

$$\begin{aligned} K_{\mathcal{Z}_1}(z) &= \pi_1^{-1} \circ I \circ J \circ \pi_1(z) \\ &= \pi_1^{-1} I \left(\frac{1}{\lambda \otimes \nu} + s \begin{pmatrix} 0 & 0 \\ 0 & v' \end{pmatrix} + O(s^2) \right) \\ &= \pi_1^{-1} \left(B\, I \left(A \left(\frac{1}{\lambda \otimes \nu} + s \begin{pmatrix} 0 & 0 \\ 0 & v' \end{pmatrix} + O(s^2) \right) B \right) A \right) \\ &= \pi_1^{-1} \left(B\, I \begin{pmatrix} 1 & 0 \\ 0 & sv' + O(s^2) \end{pmatrix} A \right), \end{aligned}$$

and the Proposition follows if we let $s \to 0$. $\square$

Now we will use the identities

$$K_{\mathcal{Z}_1} \circ J_{\mathcal{Z}_1} = I_{\mathcal{Z}_1}, \quad I_{\mathcal{Z}_1} \circ K_{\mathcal{Z}_1} = J_{\mathcal{Z}_1}.$$

Proposition 2.2. *We have $K_{\mathcal{Z}_1}(JR_{q-1}) = \mathcal{R}^1$, and thus JR_{q-1} is not exceptional for $K_{\mathcal{Z}_1}$.*

Proof. For generic s, λ, ν, v, and v' as in (2.6), we have (2.7) in the previous Proposition. Letting $s \to 0$, we see that these points are dense in $\mathcal{R}^1$, and thus $J_{\mathcal{Z}_1}\mathcal{R}^1 = \mathcal{R}^1$. Now

$$\begin{aligned} K_{\mathcal{Z}_1}(J(R_{q-1})) &= I_{\mathcal{Z}_1}(R_{q-1}) = I_{\mathcal{Z}_1}(K_{\mathcal{Z}_1}\mathcal{R}^1) \\ &= J_{\mathcal{Z}_1}(\mathcal{R}^1) = \mathcal{R}^1, \end{aligned}$$

where the second equality in the first line follows from the previous Proposition. □

§3. Construction of $\mathcal{A}^{i,j}$

We let $A_{i,j}$ denote the set of $q \times q$ matrices whose i-th row and j-th columns consist entirely of zeros. Let $\pi_2 : \mathcal{Z}_2 \to \mathbf{P}(\mathcal{M}_q)$ denote the space obtained by blowing up along all of the the centers $A_{i,j}$ for $1 \le i, j \le q$. As we discussed earlier, it will be immaterial for our purposes what order we do the blowups in. Without loss of generality, we fix our discussion on $(i,j) = (1,1)$. The set $A_{1,1}$ is equal to the set V which was introduced in the previous section. Let us use the notation

$$U = U_{1,r} = \{z \in \mathcal{M}_q : z = \begin{pmatrix} * & * \\ * & 0_{q-1} \end{pmatrix}, \ z_{1,r} = 1\} \tag{3.1}$$

for the matrices which consist of zeros except for the first row and column, and which are normalized by the entry $z_{1,r}$. With this notation and with $W_{k,\ell}$ and $V_{k,\ell}$ as in (2.1,2), we define the coordinate chart

$$\pi_2 : \mathbf{C} \times U \times V_{k,\ell} \to W_{k,\ell} \subset \mathcal{M}_q, \quad \pi_2(s,\zeta,v) = s\zeta + v = \begin{pmatrix} s\zeta & s\zeta \\ s\zeta & v \end{pmatrix}. \tag{3.2}$$

Coordinate charts of this form give a covering of $\mathcal{A}^{1,1}$, and $\{s = 0\}$ defines the set $\mathcal{A}^{1,1}$ within each coordinate chart. If $x \in \mathcal{M}_q$, then we normalize to obtain $\tilde{x} := x/x_{k,\ell} \in W_{k,\ell}$, and

$$\pi_2^{-1}(x) = (s,\zeta,v), \quad v = \tilde{x}_{[1,1]}, \ s = \tilde{x}_{1,r}, \ \zeta = (\tilde{x} - v)/\tilde{x}_{1,r}. \tag{3.3}$$

We let $K_{\mathcal{Z}_2} = \pi_2^{-1} \circ K \circ \pi_2$ denote the induced birational map on $\mathcal{Z}_2$.

Proposition 3.1. *For $1 \le r, s \le q$, $K_{\mathcal{Z}_2}(\Sigma_{r,s}) = \mathcal{A}^{s,r}$, and in particular $\Sigma_{r,s}$ is not exceptional for $K_{\mathcal{Z}_2}$.*

Proof. As was noted at the beginning of the previous section, it is no loss of generality to assume $(r,s) = (1,1)$ and $2 \le k, \ell \le q$. For generic $x \in \mathcal{M}_q$, we may use $\hat{K}$ from (1.4) and define y by

$$\hat{K}(x) = \Pi(x)\left(C_{j,i}(\frac{1}{x})\right) = y.$$

We write $\pi(\sigma,\zeta,v)=y$, and we next determine σ, ζ and v. Now let us use the notation $s=x_{1,1}$, so $\Pi(x)=s\Pi'(x)$, where Π' denotes the product of all $x_{a,b}$ except $(a,b)=(1,1)$. For $2\le i,j\le q$, we have

$$y_{i,j}=s\Pi'(x)\left(\frac{1}{s}a_{i,j}(x)+O(1)\right)$$

with $a_{i,j}(x)=(-1)^{i+j}det((1/x)_{[j,i],[1,1]})$, which gives

$$v_{i,j}=\tilde{y}_{i,j}=y_{i,j}/y_{k,\ell}=a_{i,j}(x)/a_{k,\ell}(x)+O(s),\quad 2\le i,j\le q.$$

For generic x, we may let $s\to 0$, and then the value of v approaches $(a_{i,j}(x))/a_{k,\ell}(x)$, which by (1.4) of Proposition 1.1 is just $K_{q-1}(x_{[1,1]})$, normalized at the (k,ℓ) slot.

The first row and column of $C_{j,i}(\frac{1}{x})$ do not involve the (1,1) entry of the matrix x, so $y_{1,*}$ and $y_{*,1}$ are divisible by s. By (3.3), we have $\sigma=y_{1,r}/y_{k,\ell}=O(s)$, so we see that $\sigma\to 0$ as $s\to 0$.

An element of the first row of y is given by $y_{1,j}=\Pi(x)(-1)^{j+1}det(1/x_{[j,1]})$. If we expand this determinant into minors along the top row, we have

$$y_{1,j}=\Pi(x)\sum_{2\le p\le q}(-1)^{j+1+p}det\left((1/x_{[j,1]})_{[1,p]}\right)x_{1,p}^{-1}$$

We use the notation $y_{1,*}$ and $(1/x_{1,*})$ for the vectors $(y_{1,p})_{2\le p\le q}$ and $(1/x_{1,p})_{2\le p\le q}$. Thus we find $y_{1,*}=v\,(1/x_{1,*})$. It is evident that $y_{1,1}=det(1/x_{[1,1]})$.

Now we consider the range of K near $A_{1,1}$. We have seen that $v=K_{q-1}(x_{[1,1]})$, so the values of v are dense in $V_{k,\ell}$. Now for fixed v, we see that the values of $y_{1,*}$ and $y_{*,1}$ span a $2q-2$ dimensional set. Thus, as we let the values of $x_{1,*}$ and $x_{*,1}$ range over generic values in $\mathbf{C}^{q-1}\times\mathbf{C}^{q-1}$, we see that ζ is dense in U. Thus $K_{\mathcal{Z}_2}(\Sigma_{1,1})=\mathcal{A}^{1,1}$. $\square$

§4. Construction of $\mathcal{B}^{i,j}$

For $1\le i,j\le q$, we let $U_{i,j}=\{\zeta\in\mathcal{M}_q:\zeta_{[i,j]}=0\}$ to be the set of matrices for which all entries are zero except on the i-th row and j-th column. In the construction of $\mathcal{A}^{i,j}$, we may consider $U_{i,j}$ (normalized) to be a coordinate chart in the fiber over a point of $A_{i,j}$. We define the set $B_{i,j}\subset\mathcal{A}^{i,j}\subset\mathcal{Z}_2$ to be given in local coordinates by $B_{i,j}=\{(s,\zeta,v)\in\mathcal{A}^{i,j}:s=0,\zeta_{i,j}=0\}$. This has codimension 2 in $\mathcal{Z}_2$, and we let $\pi_3:\mathcal{Z}_3\to\mathcal{Z}_2$ be the new manifold obtained by blowing up all the sets $B_{i,j}$. Let $K_{\mathcal{Z}_3}$ denote the induced birational map on $\mathcal{Z}_3$. As we have seen before, we may focus our attention on the case $(i,j)=(1,1)$. Let us use the (s,ζ,v) coordinate system (3.2) at $\mathcal{A}^{1,1}$. Let U be as in (3.1), and set $U'=\{\zeta\in U:\zeta_{1,1}=0\}$. We define the coordinate projection

$$\pi_3:\mathbf{C}\times\mathbf{C}\times U'\times V_{1,1}\to\mathbf{C}\times U\times V_{1,1},\;\;\pi(t,\tau,\xi,v)=(s,\zeta,v),\;\;s=t,\zeta=(t\tau,\xi),v=v,\quad(4.1)$$

where the notation $\zeta=(t\tau,\xi)$ means that $\zeta_{1,1}=t\tau$, and $\zeta_{a,b}=\xi_{a,b}$ for all $(a,b)\ne(1,1)$. Thus $\mathcal{B}^{1,1}$ is defined by the condition $\{t=0\}$ in this coordinate chart. Composing the two coordinate projections, $\mathcal{Z}_3\to\mathcal{Z}_2$ and $\mathcal{Z}_2\to\mathcal{M}_q$, we have

$$\pi:(t,\tau,\xi,v)\mapsto\begin{pmatrix}t^2\tau & t\xi\\ t\xi & v\end{pmatrix}=x.\qquad(4.2)$$

From (4.2), we see that $\pi^{-1}(x) = (t, \tau, \xi, v)$, where

$$\tilde{x} = x/x_{\ell,k},\ v = \tilde{x}_{[1,1]},\ t = \tilde{x}_{1,r},\ \tau = \tilde{x}_{1,1}/t^2,\ \xi_{1,j} = x_{1,j}/x_{1,r},\ 2 \le j \le q. \qquad (4.3).$$

We will use the following homogeneity property of K. If $x \in \mathcal{M}_q$, we let $\chi_t(x)$ denote the matrix obtained by multiplying the 1st row by t and then the 1st column by t, so the (1,1) entry is multiplied by t^2. It follows that $\chi_t J \chi_t = J$ and $\chi_t I \chi_t = I$, so

$$K\begin{pmatrix} \tau & \xi \\ \xi & v \end{pmatrix} = \begin{pmatrix} \tau' & \xi' \\ \xi' & v' \end{pmatrix} \text{ implies } K\begin{pmatrix} t^2\tau & t\xi \\ t\xi & v \end{pmatrix} = \begin{pmatrix} t^2\tau' & t\xi' \\ t\xi' & v' \end{pmatrix}. \qquad (4.4)$$

Proposition 4.1. *For $1 \le i, j \le q$, we have $K_{\mathcal{Z}_3}(\mathcal{B}^{i,j}) = \mathcal{B}^{j,i}$, and in particular, $\mathcal{B}^{i,j}$ is not exceptional.*

Proof. As before, we may assume that $(i, j) = (1, 1)$. A point near $\mathcal{B}^{1,1}$ may be represented in the coordinate chart (4.2) as $\pi(t, \tau, \xi, v) = \begin{pmatrix} t^2\tau & t\xi \\ t\xi & v \end{pmatrix} = x$. We define τ', ξ', and v' by the condition $K\begin{pmatrix} \tau & \xi \\ \xi & v \end{pmatrix} = \begin{pmatrix} \tau' & \xi' \\ \xi' & v' \end{pmatrix}$, so $K(x)$ is given by the right hand side of (4.4). By (4.3), the coordinates $(t'', \tau'', \xi'', v'') = \pi^{-1}K(x)$ are

$$v'' = v/v_{k,\ell},\ t'' = t\xi'_{1,r}/v'_{k,\ell},\ \tau'' = \tau(v'_{k,\ell}/\xi'_{1,r})^2.$$

From this we see that $t'' \to 0$ as $t \to 0$, which means that $K_{\mathcal{Z}_3}(\mathcal{B}^{1,1}) \subset \mathcal{B}^{1,1}$. And since K is dominant on $\mathbf{P}(\mathcal{M}_q)$, we see that $K_{\mathcal{Z}_3}(\mathcal{B}^{1,1})$ is dense in $\mathcal{B}^{1,1}$. □

Next we see how $\mathcal{A}^{i,j}$ maps under $K_{\mathcal{Z}_3}$. A point near $\mathcal{A}^{1,1}$ may be written in coordinates (3.2) as (s, ζ, v). We write K of this point in coordinates (4.1) as (t, τ, ξ, w).

Proposition 4.2. *For $1 \le i, j \le q$, we have $K_{\mathcal{Z}_3}(\mathcal{A}^{i,j}) \subset \mathcal{B}^{j,i}$. Further, $\frac{dt}{ds} \ne 0$ at generic points $(0, \zeta, v) \in \mathcal{A}^{i,j}$.*

Proof. Without loss of generality we assume $(i, j) = (1, 1)$. Let us define x and y as

$$x = \pi_2(s, \zeta, v) = \begin{pmatrix} s\zeta & s\zeta \\ s\zeta & x \end{pmatrix}, \quad y = \hat{K}(x) = \Pi(x)C\begin{pmatrix} 1 \\ x \end{pmatrix}.$$

For $2 \le h, m \le q$ there are polynomials $a_{h,m}(\zeta, v)$ and $b_{h,m}(\zeta, v)$ such that

$$y_{1,1} = s^{2q-1}a_{1,1}(\zeta, v),\ y_{1,m} = s^{2q-2}a_{1,m}(\zeta, v),\ y_{h,m} = s^{2q-3}a_{h,m}(\zeta, v) + s^{2q-2}b_{k,m}(\zeta, v).$$

We have $t = s\, a_{1,r}/a_{k,\ell} + O(s^2)$, so $dt/ds \to a_{1,r}/a_{k,\ell}$ as $s \to 0$. Thus $dt/ds \ne 0$ at generic points of $\mathcal{A}^{1,1} = \{s = 0\}$. By (4.3), we see that

$$(t, \tau, \xi, w) \to (0, a_{1,1}a_{k,\ell}/a_{1,r}^2, a_{1,*}/a_{1,r}, a_{[1,1]}/a_{k,\ell}) \in \mathcal{B}^{1,1}$$

as $s \to 0$. □

§5. Picard Group $Pic(\mathcal{Z})$

$Pic(\mathbf{P}(\mathcal{M}_q)) = \langle H \rangle$ is generated by any hyperplane H. We write $\mathcal{Z} := \mathcal{Z}_3$ and recall that each time we blow up, the exceptional (blowup) fiber gives a new basis element of the Picard group. We will work with the following basis for $Pic(\mathcal{Z})$:

$$\{H, \mathcal{R}^1, \mathcal{A}^{i,j}, \mathcal{B}^{i,j}, 1 \le i, j \le q\}. \tag{5.1}$$

Now consider the hypersurface $\Sigma_{i,j}$. Pulling this back under $\pi_1 : \mathcal{Z}_1 \to \mathbf{P}(\mathcal{M}_q)$, we find

$$\pi_1^* \Sigma_{i,j} = H_{\mathcal{Z}_1} = \Sigma_{i,j},$$

where $\Sigma_{i,j}$ on the right hand side denotes the strict transform $\pi^{-1}\Sigma_{i,j}$. The equality between the strict and total transforms follows because the indeterminacy locus $\mathcal{I}(\pi_1^{-1}) = R_1$ is not contained in $\Sigma_{i,j}$. On the other hand, if we define

$$T_{i,j} := \{(a, b) : a = i \text{ or } b = j\} \tag{5.2},$$

then $\Sigma_{i,j}$ contains $A_{a,b}$ exactly when $(a, b) \in T_{i,j}$. Thus, pulling back under $\pi_2 : \mathcal{Z}_2 \to \mathcal{Z}_1$, we have

$$\pi_2^* \Sigma_{i,j} = H_{\mathcal{Z}_2} = \Sigma_{i,j} + \sum_{(a,b)\in T_{i,j}} \mathcal{A}^{a,b}.$$

We will next pull this back under $\pi_3 : \mathcal{Z}_3 \to \mathcal{Z}_2$. For this, we note that $B_{a,b} \subset \mathcal{A}^{a,b}$, and in addition $B_{i,j} \subset \Sigma_{i,j}$. Rearranging our answer, we have:

$$\Sigma_{i,j} = H_{\mathcal{Z}} - \mathcal{B}^{i,j} - \sum_{(a,b)\in T_{i,j}} \left(\mathcal{A}^{a,b} + \mathcal{B}^{a,b}\right). \tag{5.3}$$

Proposition 5.1. *The class of JR_{q-1} in $Pic(\mathcal{Z})$ is given in the basis (5.1) by*

$$JR_{q-1} = (q^2 - q)H - (q-1)\mathcal{R}^1 - (2q-3)\sum_{a,b} \mathcal{A}^{a,b} - (2q-2)\sum_{a,b} \mathcal{B}^{a,b}. \tag{5.4}$$

Proof. The polynomial $P(x) := \Pi(x)det(\frac{1}{x})$, analogous to (1.4), is irreducible and has degree $q^2 - q$. Thus $JR_{q-1} = \{P = 0\} = (q^2 - q)H$ in $Pic(\mathbf{P}(\mathcal{M}_q))$. Now we pull this back under the coordinate projection π_1 in (2.3). That is, we evaluate $P(x)$ for $x = \pi_1(s, \lambda, \nu, v)$). For $s = 0$ and generic λ, ν, and v, the entries of $x = \lambda \otimes \nu + sv$ are nonzero, so $\Pi(x) \ne 0$. We will show $det(\frac{1}{x}) = \alpha s^{q-1} + \cdots$, where $\alpha \ne 0$ for generic λ, ν, and v. By (2.7), we must evaluate $det(M)$ with $M = \lambda^{-1} \otimes \nu^{-1} + sv' + O(s^2)$. Now we may do elementary row and column operations, such as add $\lambda_j^{-1}\nu$ to the jth row, which do not change the determinant. In this way, we see that $det(M)$ is equal to $det \begin{pmatrix} 1 & 0 \\ 0 & sv' + O(s^2) \end{pmatrix} = \alpha s^{q-1} + \cdots$. This means that

$$(q^2 - q)H = \pi_1^*(JR_{q-1}) = JR_{q-1} + (q-1)\mathcal{R}^1 \in Pic(\mathcal{Z}_1).$$

Now we bring this back to $\mathcal{Z}_2$ by pulling back under the projection π_2 defined in (3.2). In this case, we have $\Pi(\pi_2(s,\zeta,v)) = \alpha s^{2q-1} + \cdots$, where $\alpha = \alpha(\zeta, v) \neq 0$ for generic ζ and v. On the other hand, we have $det \begin{pmatrix} s^{-1}\zeta^{-1} & s^{-1}\zeta^{-1} \\ s^{-1}\zeta^{-1} & v^{-1} \end{pmatrix} = s^{-2}\beta + \cdots$, and $\beta(\zeta, v) \neq 0$ at generic points. Thus $P(\pi_2(s,\zeta,v)) = cs^{q-3} + \cdots$, which gives the coefficient $2q-3$ for each $\mathcal{A}^{i,j}$:

$$(q^2-q)H = JR_{q-1} + (q-1)\mathcal{R}^1 + (2q-3)\sum_{i,j}\mathcal{A}^{i,j} \in Pic(\mathcal{Z}_2).$$

Pulling back to $\mathcal{Z}_3$ is similar, except that $\Pi(\pi_3(t,\tau,\xi,v) = \alpha t^{2q} + \cdots$. Thus we obtain the coefficient $2q-2$ for $\mathcal{B}^{i,j}$ in (5.4). □

§6. The induced map $K_{\mathcal{Z}}^*$ on $Pic(\mathcal{Z})$

We define the pullback map on functions by composition $K_{\mathcal{Z}}^*\varphi := \varphi \circ K_{\mathcal{Z}}$. We may apply $K_{\mathcal{Z}}^*$ to local defining functions of a divisor, and since $K_{\mathcal{Z}}$ is well defined off the indeterminacy locus, which has codimension ≥ 2, $K_{\mathcal{Z}}^*$ induces a well-defined pullback map on $Pic(\mathcal{Z})$.

Proposition 6.1. *$K_{\mathcal{Z}}^*$ maps the basis (5.1) according to:*

$$\begin{aligned} H &\mapsto (q^2-q+1)H - (q-2)\mathcal{R}^1 - \sum_{a,b}\left((2q-3)\mathcal{A}^{a,b} + (2q-2)\mathcal{B}^{a,b}\right) \\ \mathcal{R}^1 &\mapsto (q^2-q)H - (q-1)\mathcal{R}^1 - \sum_{a,b}\left((2q-3)\mathcal{A}^{a,b} + (2q-2)\mathcal{B}^{a,b}\right) \\ \mathcal{A}^{i,j} &\mapsto H - \mathcal{B}^{j,i} - \sum_{(a,b)\in T_{i,j}}\left(\mathcal{A}^{a,b} + \mathcal{B}^{a,b}\right) \\ \mathcal{B}^{i,j} &\mapsto \mathcal{A}^{j,i} + \mathcal{B}^{j,i} \end{aligned} \tag{6.1}$$

Proof. Let us start with $\mathcal{R}^1$. By §2, $K_{\mathcal{Z}}|_{JR_{q-1}}$ is dominant as a map to $\mathcal{R}^1$. Since $K_{\mathcal{Z}}$ is birational, it is a local diffeomorphism at generic points of JR_{q-1}. Thus we have $K_{\mathcal{Z}}^*(\mathcal{R}^1) = JR_{q-1}$, so the second line in (6.1) follows from Proposition 5.1.

Similarly, since $K_{\mathcal{Z}}|_{\Sigma_{i,j}}$ is a dominant map to $\mathcal{A}^{j,i}$, we have $K_{\mathcal{Z}}^*(\mathcal{A}^{i,j}) = \Sigma_{j,i}$, and the third line of (6.1) follows from (5.3).

In the case of $\mathcal{B}^{i,j}$, we know from §4 that $K_{\mathcal{Z}}^{-1}\mathcal{B}^{i,j} = \mathcal{A}^{j,i} \cup \mathcal{B}^{j,i}$. Thus $K_{\mathcal{Z}}^*\mathcal{B}^{i,j} = \lambda\mathcal{A}^{j,i} + \mu\mathcal{B}^{j,i}$ for some integer weights λ and μ. Again, since $K_{\mathcal{Z}}$ is birational, and $K_{\mathcal{Z}}|_{\mathcal{B}^{i,j}}$ is a dominant map to $\mathcal{B}^{j,i}$, we have $\mu = 1$. Proposition 4.2 gives us $\lambda = 1$.

Finally, set $h(x) = \sum_{i,j} a_{i,j}x_{i,j}$, and let $H = \{h = 0\}$ be a hyperplane. The pullback is given by the class of $\{h\hat{K}(x) = 0\} = \sum_{i,j} a_{i,j}\hat{K}_{i,j}(x) = 0$, where $\hat{K}$ is given by (1.4). Pulling back h is similar to the situation in Proposition 5.1, where we pulled back the function $P(x)$. The difference is that instead of working with $det(\frac{1}{x})$ we are working with all of the $(q-1)\times(q-1)$ minors. By Proposition 1.1, we have $K^*H = (q^2-q+1)H \in Pic(\mathbf{P}(\mathcal{M}_q))$. Next we will move up to $\mathcal{Z}_1$ by pulling back under π_1 and finding the multiplicity of $\mathcal{R}^1$. We consider $h\hat{K}\pi_1(s,\lambda,\nu,v)$, and we recall the matrix M from the proof of Proposition 5.1. We see that each $(q-1)\times(q-1)$ minor of M is either $O(s^{q-1})$ or $O(s^{q-2})$. Thus for a generic hyperplane, the order of vanishing is $q-2$, so we have

$$(q^2-q+1)H = K^*H + (q-2)\mathcal{R}^1 \in Pic(\mathcal{Z}_1).$$

Next, to move up to $\mathcal{Z}_2$, we look at the order of vanishing of $h\hat{K}\pi_2(s,\zeta,v)$ in s. Again $\Pi(\pi_2(s,\zeta,v)) = \alpha s^{2q-1}+\cdots$. The $(q-1)\times(q-1)$ minors of $\begin{pmatrix} s^{-1}\zeta^{-1} & s^{-1}\zeta^{-1} \\ s^{-1}\zeta^{-1} & v^{-1} \end{pmatrix}$ which are most singular at $s=0$ behave like $s^{-2}\beta+\cdots$. Thus for generic coefficients $a_{i,j}$ we have vanishing to order $2q-3$ in s, and so $2q-3$ is the coefficient for each $\mathcal{A}^{i,j}$ as we pull back to $Pic(\mathcal{Z}_2)$. Coming up to $\mathcal{Z}_3=\mathcal{Z}$, we pull back under π_3, and the calculation of the multiplicity of $\mathcal{B}^{i,j}$ is similar. This gives the first line in (6.1). □

Proposition 6.2. *The characteristic polynomial of the transformation (6.1) is*

$$P(\lambda)Q(\lambda)^{q-1}(\lambda-1)^{q^2-q+2}(\lambda+1)^{q^2-3q+2},$$

where $P(\lambda)=\lambda^2-(q^2-4q+2)\lambda+1$ *and* $Q=(\lambda^2+1)^2-(q-2)^2\lambda^2$.

Proof. We will exhibit the invariant subspaces of $Pic(\mathcal{Z})$ which correspond to the various factors of the characteristic polynomial. First, we set $\mathcal{A}:=\sum\mathcal{A}^{k,\ell}$ and $\mathcal{B}:=\sum\mathcal{B}^{k,\ell}$, where we sum over all k and ℓ, and we set $S_1=\langle H,\mathcal{R}^1,\mathcal{A},\mathcal{B}\rangle$. By (6.1), S_1 is $K^*_{\mathcal{Z}}$-invariant, and the characteristic polynomial of $K^*_{\mathcal{Z}}|_{S_1}$ is seen to be $P(\lambda)(\lambda-1)^2$.

Next, if $i<j$, then we set $\alpha_{i,j}=\mathcal{A}^{i,i}+\mathcal{A}^{j,j}-(\mathcal{A}^{i,j}+\mathcal{A}^{j,i})$, and similarly for $\beta_{i,j}$, using the $\mathcal{B}^{k,\ell}$. Then by (6.1), $S_{i,j}:=\langle\alpha_{i,j},\beta_{i,j}\rangle$ is invariant, and the characteristic polynomial of $K^*_{\mathcal{Z}}|_{S_{i,j}}$ is $(\lambda-1)^2$.

Similarly, if $i<j<k$, we set $\alpha_{i,j,k}=\mathcal{A}^{i,j}+\mathcal{A}^{j,k}+\mathcal{A}^{k,i}-\left(\mathcal{A}^{j,i}+\mathcal{A}^{k,j}+\mathcal{A}^{i,k}\right)$ and define $\beta_{i,j,k}$ similarly. Then the 2-dimensional subspace $S_{i,j,k}:=\langle\alpha_{i,j,k},\beta_{i,j,k}\rangle$ is invariant, and the characteristic polynomial of $K^*_{\mathcal{Z}}|_{S_{i,j,k}}$ is $(\lambda+1)^2$.

Finally, for each i, we consider the row and column sums $\mathcal{A}_{r_i}=q\sum_j\mathcal{A}^{i,j}-\mathcal{A}$, $\mathcal{A}_{c_j}=q\sum_i\mathcal{A}^{i,j}-\mathcal{A}$, and we make the analogous definition for $\mathcal{B}_{r_i}$ and $\mathcal{B}_{c_j}$. The 4-dimensional subspace $\langle\mathcal{A}_{r_i},\mathcal{A}_{c_i},\mathcal{B}_{r_i},\mathcal{B}_{c_i}\rangle$ is invariant and yields the factor $Q(\lambda)$. These invariant subspaces span $Pic(\mathcal{Z})$, and the product of these factors gives the characteristic polynomial stated above. □

Proof of the Theorem. The spectral radius of $K^*_{\mathcal{Z}}$ is the modulus of the largest root of the characteristic polynomial, which is given in Proposition 6.2. By inspection, the largest root of the characteristic polynomial is the largest root of $P(\lambda)$. The spectral radius of $K^*_{\mathcal{Z}}$ is an upper bound for $\delta(K)$. On the other hand, it was shown in [BV] that this same number is also a lower bound for $\delta(K)$, so the Theorem is proved. □

Remark Let us conclude with a discussion of the exceptional cases $q=3$ and 4. The proof above shows that $\delta(K_q)=1$ if $q=3$ or 4. In fact one can show that $(K^*_{\mathcal{Z}})^n=(K^n_{\mathcal{Z}})^*$ for all q. To determine the degree growth, we need to know what $(K^*_{\mathcal{Z}})^n$ does to H, and so we consider the restriction of $K^*_{\mathcal{Z}}$ to the subspace S_1 which is defined in the proof of Proposition 6.2. When $q=3$, the non-diagonal part of the Jordan canonical form $K^*_{\mathcal{Z}}|_{S_1}$ is a 2×2 Jordan block with eigenvalue 1. Thus the degree of K^n grows linearly in n. When $q = 4$, $K^*_{\mathcal{Z}}|_{S_1}$ is a 4×4 Jordan block with eigenvalue 1, and in this case the degree of K^n grows like the cube of n.

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